# Analytical Spectrum Isomorphism of Noncommutative Harmonic Oscillator and Charged Particle in Magnetic Field

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**Abstract.** The comparison of the Hamiltonians of a noncommutative isotropic harmonic oscillator and a charged particle in a uniform magnetic field in symmetric and Landau gauges are analysed to study the specific conditions under which these two models are indistinguishable. The Hamiltonian of the noncommutative isotropic harmonic oscillator is stated by applying Bopp shift transformation. The result shows that the two systems are isomorphic up to the similar values of quantum numbers $n_r$ and $m_l$ with the product of electric charge and magnetic field to be $qB = eB > 0$ for both gauge choices respectively. In addition, there is an extra requirement for Landau gauge where the noncommutative oscillator has to lose a single spatial degree of freedom. It also needs to be parametrized by a factor $\zeta$ to be determined for their Hamiltonians to be consistent with each other.

**Keywords:** Landau levels, isotropic harmonic oscillator, noncommutative quantum mechanics.

## I. INTRODUCTION

Recently, noncommutative field theories and their derivatives have been a subject of intense research. We refer to [1, 2, 3] as reading materials to provide some historical background and reviews on the subject over the past few years. Since their establishment, they have invaded various domains of theoretical physics and have effectively continued to evolve and meet the mathematical requirements of various situations. These include string theory [4, 5, 6], Yang - Mills theory whereby some of its forms is a noncommutative gauge theory [7, 8, 9] and condensed matter theory most notably, quantum Hall effect [10].

A simple insight on the role of noncommutativity in field theory can be obtained by studying the one particle sector, which prompted an interest in the study of noncommutative quantum mechanics (NCQM). The deformation of space due to noncommutativity can be expressed by the commutation relations of Hermitian operators as shall be seen later in Section III. Thus, several authors have solved many related problems for example, hydrogen atom [11], central potential [12], Aharonov-Bohm effect [13], Ahoronov - Casher effect [14], Klein-Gordon oscillators [15], the Landau problem [16], and the list goes on.

In [17], the relationship between magnetic field and noncommutativity parameter has been established ($\theta = \frac{4\hbar}{qB}$) for the noncommutative quantum mechanics and the usual Landau problem to be equivalent theories in the lowest Landau levels in symmetric gauge. Landau problem here refers to the quantization of the cyclotron orbits of charged particles in a uniform magnetic field. These charged particles can only occupy orbits with discrete, equidistant energy values, called Landau levels. A further generalization of this relation was then made in [18] in the context of relativistic quantum mechanics. In this work, we would like to argue that the isomorphism between one of the problems of noncommutative quantum mechanics i.e, isotropic harmonic oscillator and charged particle in magnetic field can also exist in higher-order Landau levels. In fact, it occurs to all Landau levels for both symmetric and Landau gauges provided that their specific conditions to be discussed are satisfied.

The paper is organized as follows. Section II is devoted to the description of Hamiltonian of charged particle in magnetic field for both gauges respectively. Section III focuses on finding the solution to the eigenvalue problem of a noncommutative isotropic harmonic oscillator by virtue of comparison of its Hamiltonian and that of charged particle in magnetic field. In the final section, we state our conclusion.

## II. CHARGED PARTICLE IN MAGNETIC FIELD

In the presence of a magnetic field $\hat{B}$, the canonical momentum $\hat{p}$ is shifted by the magnetic vector potential $\hat{A}$ to give a Hamiltonian of the form

$$\hat{H}_L = \frac{(\hat{p}-q\hat{A})^2}{2\mu}, \tag{1}$$

where $q$ and $\mu$ are the charge and the mass of the particle of interest, and $\hat{p} - q\hat{A}$ is the kinetic momentum operator. Hereafter, we adopt $q = -e$ which denotes the charge of an electron. Note that in the presence of a magnetic field, $\hat{p} - q\hat{A}$ represents the true momentum of the particle rather than $\hat{p}$ and is the result of minimal coupling rule. In this problem, we consider an electron to be freely moving in a two-dimensional system in a uniform magnetic field, $\hat{B} = B\hat{z}$. For such a magnetic field, there are two common gauge choices which will be discussed in greater detail shortly.

### Symmetric Gauge

For the first gauge choice i.e, the symmetric gauge, the vector potential can be written as

$$\hat{A}_x = -\frac{1}{2}B\hat{y}, \quad \hat{A}_y = \frac{1}{2}B\hat{x}. \tag{2}$$

Then, by substituting (2) into (1),

$$\hat{H}_L^{(sym)} = \frac{1}{2\mu}\left[\left(\hat{p}_x + \frac{qB}{2}\hat{y}\right)^2 + \left(\hat{p}_y - \frac{qB}{2}\hat{x}\right)^2\right]. \tag{3}$$

Note that we define the z-axis such that $qB > 0$. For the case of an electron where $q < 0$, it means that the magnetic field is along the negative $z$-axis. Alternatively, $qB > 0$ is also true for a proton where $q > 0$ when the magnetic field is along the positive $z$-axis. Hence, we will now adopt $qB = eB$ instead where $eB$ is also greater than 0. After a few algebraic manipulation steps,

$$\widehat{H}_L^{(sym)} = \frac{1}{2\mu}\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{e^2B^2}{8\mu}\left(\hat{x}^2 + \hat{y}^2\right) - \frac{eB}{2\mu}\left(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x\right), \tag{4}$$

where $\hat{L}_z \equiv \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ is the $z$-component of the orbital angular momentum operator and $\omega_c = \frac{eB}{\mu}$ represents the cyclotron frequency in SI units. However, we want to highlight that in some literature such as in [19, 25], the Hamiltonian and the cyclotron frequency can also be written in *Gaussian* units. Despite the difference of units, both are valid representations of the Hamiltonian for the system. In this work, the Hamiltonian which considers the SI units of the cyclotron frequency will be considered.

Returning to our initial problem of calculating the eigenstates and eigenvalues of (4), by applying a few change of variables and orthogonality relation, the solution to the time-independent Schrödinger equation is

$$\begin{aligned}\Psi_{n_r,m_l}(r,\varphi) = \; & \frac{1}{\sqrt{2\pi}}\left(\frac{eB}{\hbar}\right)^{\frac{1}{2}}\sqrt{\frac{n_r!}{(n_r+|m_l|)!}}\left(\frac{eB}{2\hbar}r^2\right)^{\frac{|m_l|}{2}} \\ & \exp\left(-\frac{eB}{4\hbar}r^2\right)L_{n_r}^{|m_l|}\left(\frac{eB}{2\hbar}r^2\right)e^{im_l\varphi}.\end{aligned} \tag{5}$$

where $n_r$ and $m_l$ are radial and angular momentum quantum numbers respectively, $L_{n_r}^{|m_l|}$ is the generalized Laguerre polynomial [20], and the result is in agreement with [21]. As for the eigenvalues, they are given as

$$E_{n_r,m_l} = (2n_r + |m_l| + 1)\hbar\frac{\omega_c}{2} - m_l\hbar\frac{\omega_c}{2}. \tag{6}$$

The first term mimics the energy eigenvalues of the standard two-dimensional harmonic oscillator (with the frequency $\frac{\omega_c}{2}$ instead). The second term comes from the coupled momentum-gauge term that is manifested in the Hamiltonian after expanding the kinetic momentum operator. This term represents the interaction between the canonical momenta and the magnetic field, and what distinguishes this problem from a standard planar isotropic oscillator [21].

Now, the interesting question that could be pondered is, what if we choose to use $qB = -eB$ instead? At first, this may look trivial but a careful calculation reveals that the resulting energy levels are actually different from the case that we obtain previously. In fact, it can be shown that

$$E_{n_r,m_l} = (2n_r + |m_l| + 1)\hbar\frac{\omega_c}{2} + m_l\hbar\frac{\omega_c}{2}. \tag{7}$$

while the eigenfunction maintains the same form as in (5). Notice that the sign of the second term is different than before which implies that, the mere assumption that we make on the product, $qB$ can actually change the pattern of degeneracy. We refer to [21] for a detailed derivation of the eigenstates and eigenenergies obtained when $qB = -eB$.

## Landau Gauge

For the second gauge choice i.e, the Landau gauge, the vector potential can be written as

$$\hat{A}_x = -B\hat{y}, \hat{A}_y = 0, \tag{8}$$

or $\hat{A}_x = 0, \hat{A}_y = B\hat{x}$ to represent the first and second Landau gauges respectively. From here thereon, we are only going to focus our attention on the former, as the discussion that follows will also be applied to the latter gauge in a similar manner to the former gauge but with a minute difference as shall be pointed out later. Then by substituting (8) into (1)

$$\hat{H}_L^{(Lan)} = \frac{1}{2\mu}\left[(\hat{p}_x + qB\hat{y})^2 + \left(\hat{p}_y\right)^2\right]. \tag{9}$$

Just like in the symmetric gauge, we will define the $z$-axis such that $qB > 0$ and hence, $qB = eB$. After a few algebraic manipulation steps,

$$\hat{H}_L^{(Lan)} = \frac{1}{2\mu}\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{e^2B^2}{2\mu}\hat{y}^2 + \frac{eB}{\mu}\hat{y}\hat{p}_x. \tag{10}$$

The derivation of the solutions of the Schrödinger equation involving the Hamiltonian in (10) is very well known as it has the mathematical structure of a shifted harmonic oscillator [22] and thus, will not be repeated here. We then obtain

$$\Psi_{n_y,k_x}(x,y) = \frac{1}{\sqrt{2^{n_y}n_y!}}\left(\frac{eB}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left[-\frac{eB}{2\hbar}\left(y + \frac{\hbar k_x}{eB}\right)^2\right] H_n\left[\frac{eB}{\hbar}\left(y + \frac{\hbar k_x}{eB}\right)\right]e^{ik_x x}, \tag{11}$$

where $H_n$ is the Hermite polynomial. This method of solution has introduced two quantum numbers; $k_x$, a real number, and $n_y$, a non-negative integer. The latter quantum number counts the number of nodes of the probability density function. On the other hand, the eigenvalues are

$$E_{n_y,k_x} = \left(2n_y + 1\right)\hbar\frac{\omega_c}{2}, \tag{12}$$

Note the lack of dependence on the quantum number $k_x$. Consequently, the energies are infinitely degenerate for every non-negative integer $n_y$. If however, we choose to take the second Landau gauge, the energy eigenvalues will also take the form of (12). In regard to the energy eigenstates, we need to change the sign of the shifted term of the position operator of the harmonic oscillator to have $-\frac{\hbar k_y}{eB}$ with quantum numbers, $k_y$ and $n_x$. In contrast to our result, in [21], an assumption of $qB = -eB$ is analysed instead. The only difference of our results with those in [21] is, we need to change the sign of the spatial shift of the eigenfunction of the oscillator in both Landau gauges while having the same energy.

## III. NONCOMMUTATIVE ISOTROPIC HARMONIC OSCILLATOR

The Hamiltonian of a particle of mass $m$ which oscillates with an angular frequency $\omega$ under the influence of an isotropic harmonic oscillator potential in the noncommutative space is formulated as

$$\hat{H}_{NC} = \frac{1}{2m}(\hat{\mathbf{p}}_1^2 + \hat{\mathbf{p}}_2^2) + \frac{1}{2}m\omega^2(\hat{\mathbf{x}}_1^2 + \hat{\mathbf{x}}_2^2). \tag{13}$$

There are various formulations of quantum mechanics on noncommutative Moyal phase spaces that can be used to deal with related problems such as the above. Among them include canonical, path-integral, Weyl-Wigner, and systematic formulations [23]. We will utilize canonical formulation in our work. Then, by applying Bopp shift transformation, the non-commuting coordinates can be expressed in terms of commuting coordinates in the following form

$$\hat{\mathbf{x}}_i = \hat{x}_i - \frac{\theta_{ij}}{2\hbar}\hat{p}_j, \qquad \hat{\mathbf{p}}_i = \hat{p}_i. \tag{14}$$

We can see from the above that $\hat{\mathbf{p}}_i$ maps onto itself. We often take $\theta_{ij} = \theta\varepsilon_{ij}$ where $\theta$ is the constant, frame-dependent parameters and $\varepsilon_{ij}$ is the Levi-Civita symbol normalized by $\varepsilon_{12} = 1$ [24]. The commutators involving position and momentum in (14) are $[\hat{\mathbf{x}}_i, \hat{\mathbf{x}}_j] = i\theta_{ij}$, $[\hat{\mathbf{p}}_i, \hat{\mathbf{p}}_j] = 0$ and $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$. Due to the transformation, the new variables will satisfy the usual commutation relations in standard quantum mechanics as shown below

$$\left[\hat{x}_i, \hat{x}_j\right] = 0, \qquad \left[\hat{x}_i, \hat{p}_j\right] = i\hbar\delta_{ij}, \qquad \left[\hat{p}_i, \hat{p}_j\right] = 0.$$

With the new variables defined in (14), the Hamiltonian in the ordinary commutative space will then be

$$\hat{H}_{NC} = \frac{1}{2m}[\hat{p}_1^2 + \hat{p}_2^2] + \frac{1}{2}m\omega^2\left[\left(\hat{x}_1 - \frac{\theta}{2\hbar}\hat{p}_2\right)^2 + \left(\hat{x}_2 + \frac{\theta}{2\hbar}\hat{p}_1\right)^2\right]. \tag{15}$$

After a few algebraic manipulation steps, setting $\hat{x}_1 = \hat{x}$ and $\hat{x}_2 = \hat{y}$, we obtain

$$\hat{H}_{NC} = \left(\frac{1}{2m} + \frac{m\omega^2\theta^2}{8\hbar^2}\right)\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{1}{2}m\omega^2(\hat{x}^2 + \hat{y}^2) - \frac{\theta}{2\hbar}m\omega^2\hat{L}_z. \tag{16}$$

Then, rearrange (16) by introducing new effective mass and frequency so that the first two terms have the form of the Hamiltonian of an isotropic harmonic oscillator as follows

$$\hat{H}_{NC} = \frac{1}{2M}\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{1}{2}M\Omega^2(\hat{x}^2 + \hat{y}^2) - \frac{\theta}{2\hbar}M\Omega^2\hat{L}_z, \tag{17}$$

where

$$M = \frac{m}{1 + \frac{m^2\omega^2\theta^2}{4\hbar^2}}; \qquad \Omega = \omega\sqrt{1 + \frac{m^2\omega^2\theta^2}{4\hbar^2}}. \tag{18}$$

The definition of the parameters above are also used in [26, 27]. We shall now address the main issue in this paper: can we find the energy eigenvalues and eigenstates of the noncommutative oscillator from the analytical spectrum of charged particle in magnetic field? This shall be explored in the upcoming subsections.

### Symmetric Gauge with Noncommutativity

In this section, we will elaborate the equivalence of analytical spectrum that exists between noncommutative oscillator and charged particle in magnetic field for the symmetric gauge. We can naively compare the Hamiltonian of the two systems (4) and (17), and then try to infer from there. Thus, we have

$$\frac{m}{1 + \frac{m^2\omega^2\theta^2}{4\hbar^2}} = \mu, \qquad \frac{1}{2}m\omega^2 = \frac{e^2B^2}{8\mu}, \qquad \frac{\theta}{2\hbar}m\omega^2 = \frac{eB}{2\mu}. \tag{19}$$

Note that, the set of equations that we have from the comparison of the coefficients of the Hamiltonians needs to be true simultaneously. In other words, (19) altogether has to be consistent but, that is not the case here. For the two systems to be isomorphic, we need to invoke some mathematical scheme by redefining $M$ and $\Omega$ in other terms instead of $m$ and $\omega$. This is allowable as $M$ and $\Omega$ are respectively the parameters that characterize the effective mass and frequency of the oscillator in commutative space. With reference to [17] which is generalized for any noncommutative quantum mechanical problem in central field, we are going to let the effective mass and frequency to be

$$M = \frac{2\hbar^2}{\theta^2}, \qquad \Omega = \frac{\theta}{\hbar}, \tag{20}$$

which are $\theta$-dependent. Due to (20), the resulting Hamiltonian of the noncommutative oscillator is the following,

$$\hat{H}_{NC} = \frac{\theta^2}{4\hbar^2}\left(\hat{p}_x{}^2 + \hat{p}_y{}^2\right) + (\hat{x}^2 + \hat{y}^2) - \frac{\theta}{\hbar}\hat{L}_z. \tag{21}$$

Realize that by imposing the oscillator to have the new effective mass and frequency in terms of $\theta$ alone, the Hamiltonian (21) lack of parameters permits an opportunity of introducing a factor to make the two systems consistent. By multiplying each of the coefficients of the Hamiltonian by a factor $\zeta$ to be determined, then

$$\hat{H}_{NC}^{(sym)} = \frac{\zeta\theta^2}{4\hbar^2}\left(\hat{p}_x{}^2 + \hat{p}_y{}^2\right) + \zeta(\hat{x}^2 + \hat{y}^2) - \frac{\zeta\theta}{\hbar}\hat{L}_z. \tag{22}$$

Now, by comparing the Hamiltonian of the charged particle in magnetic field for symmetric gauge and the newly-defined Hamiltonian of the noncommutative oscillator in (4) and (22), we have

$$\frac{1}{2\mu} = \frac{\zeta\theta^2}{4\hbar^2}, \qquad \frac{e^2B^2}{8\mu} = \zeta, \qquad \frac{eB}{2\mu} = \frac{\zeta\theta}{\hbar}. \tag{23}$$

Unlike the comparison from (19), a close inspection reveals that (23) is altogether consistent and in accordance with [17]. As a result, together with factor $\zeta$, the effective mass and frequency of the oscillator will then be equal to those of charged particle in magnetic field as

$$M = \frac{2\hbar^2}{\zeta\theta^2}; \qquad \Omega = \frac{\zeta\theta}{\hbar}, \tag{24}$$

By calculation, we can evaluate that when the charged particle in magnetic field has $\mu = m_e = 9.109 \times 10^{-31}$ and $\omega_c = \frac{eB}{\mu} = 2.110 \times 10^{12}$, then the oscillator would have $M = 4.620 \times 10^{-37}$ and $\Omega = 2.081 \times 10^{18}$ due to (20). Since we have established the relationship above, we can always examine different mass and frequency of either of the systems and be able to find the same parameters of the other system studied for them to be isomorphic.

Let us contemplate the effect of constant $B$ and $\theta$ on the respective systems. In the quantum Hall effect experiments, the magnetic fields are typically about $B = 12T$ [17, 27, 28]. Using (23), we can show that

$$\theta = \frac{4\hbar}{eB} = 2.195 \times 10^{-16}, \tag{25}$$

in $m^2$ or it can also be written as $\theta = 0.2195 \times 10^{-11}$ in $cm^2$ which is in conformance with [17]. For this value of $\theta$, one cannot distinguish between noncommutative quantum mechanics (in this case, it would be the noncommutative isotropic harmonic oscillator) and the usual Landau problem (i.e, Landau levels from the analytical spectrum of charged particle in magnetic field).

Since we have determined the value of $\theta$, we should also compute the value of the controlling factor $\zeta$, that is

$$\zeta = 5.071 \times 10^{-7}, \tag{26}$$

which implies that just like $\theta$, there is a dependence on the charge of the particle and the magnetic field which acts on it in the resulting Landau levels with an additional dependence on the mass of the particle.

Hence, since their Hamiltonians are now similar, analysing Landau levels is tantamount to analysing the noncommutative oscillator. Let us revisit the energy eigenvalues of Landau levels in symmetric gauge for $qB = eB > 0$. They are

$$E_{n_r,m_l} = (2n_r + |m_l| + 1)\hbar\frac{eB}{2\mu} - m_l\hbar\frac{eB}{2\mu}. \tag{27}$$

For the noncommutative oscillator, we can express (27) in terms of $\theta$ and $\zeta$ as follows

$$E_{n_r,m_l} = (2n_r + |m_l| + 1)\zeta\theta - m_l\zeta\theta. \tag{28}$$

Therefore, we conclude that the noncommutative oscillator and Landau levels are isomorphic provided that the pair of quantum numbers $n_r$ and $m_l$ of both systems have the same values. Besides, the Hamiltonian of the noncommutative oscillator has to be parametrized by the controlling factor $\zeta$ so that it is consistent with the Hamiltonian of Landau levels in symmetric gauge. In regards to the wavefunctions of Landau levels, we recall that

$$\Psi_{n_r,m_l}(r,\varphi) = \frac{1}{\sqrt{2\pi}}\left(\frac{eB}{\hbar}\right)^{\frac{1}{2}}\sqrt{\frac{n_r!}{(n_r+|m_l|)!}}\left(\frac{eB}{2\hbar}r^2\right)^{\frac{|m_l|}{2}}\exp\left(-\frac{eB}{4\hbar}r^2\right)L_{n_r}^{|m_l|}\left(\frac{eB}{2\hbar}r^2\right)e^{im_l\varphi}. \tag{29}$$

For the noncommutative oscillator, again by expressing (29) in terms of $\theta$ and $\zeta$, we get

$$\Psi_{n_r,m_l}(r,\varphi) = \frac{1}{\sqrt{2\pi}}\left(\frac{4}{\theta}\right)^{\frac{1}{2}}\sqrt{\frac{n_r!}{(n_r+|m_l|)!}}\left(\frac{2}{\theta}r^2\right)^{\frac{|m_l|}{2}}\exp\left(-\frac{1}{\theta}r^2\right)L_{n_r}^{|m_l|}\left(\frac{2}{\theta}r^2\right)e^{im_l\varphi}. \tag{30}$$

With the above results, we can find the analytical spectrum of the noncommutative oscillator in the context of Landau levels in symmetric gauge. However, the results obtained are by no means representing the general spectrum of the solution of the noncommutative oscillator but rather specific to the case when $M$ and $\Omega$ are $\theta$-dependent as designated in (24). We refer to [30, 31, 32] as reading materials that deliver intricate research on the spectrum of the noncommutative oscillator using path integral formulation.

The same question as raised before can still be asked here. What if we impose the following condition, $qB = -eB$? We will encounter a situation where the comparison cannot be made as

$$-\frac{\theta}{2\hbar} = \frac{eB}{2\mu}, \tag{31}$$

which is clearly false as the left-hand side is negative while the right-hand side is positive. This is the result when comparing the third term of their Hamiltonians. One might argue that why do we not account for this with a new $\zeta$. The problem is, the $\zeta$-term is also present in the first and second terms of the Hamiltonian of the noncommutative oscillator. If we change them, it will generate a false statement like in (31). Hence, we can say that this is also a condition for isomorphism. We want to emphasize again that the results of the energy spectra and wavefunctions of the noncommutative oscillator that we obtain here are applied only when $M$ and $\Omega$ are $\theta$-dependent which is the limitation of this formalism.

## Landau Gauge with Noncommutativity

In this section, we will elaborate the equivalence of analytical spectrum that exists between noncommutative oscillator and charge particle in magnetic field for the Landau gauge. Realize that unlike the previous section, we cannot directly compare the coefficients of their Hamiltonians as their variables or operators are clearly different from one another. Nonetheless, an assumption can be made for each to account for this. For the first Landau gauge, assume that the position operator, $\hat{x}$ to be the zero operator, $\hat{0}$. The same supposition goes to the second Landau gauge as well but with $\hat{y}$. Hence, the correlation can then be made as follows,

$$\frac{m}{1+\frac{m^2\omega^2\theta^2}{4\hbar^2}} = \mu, \qquad \frac{1}{2}m\omega^2 = \frac{e^2B^2}{2\mu}, \qquad \frac{\theta}{2\hbar}m\omega^2 = \frac{eB}{\mu}. \tag{32}$$

Notice that the set of equations in (32) is reminiscent to the one in symmetric gauge and therefore, we will not repeat the discussion here. However we will show how their Hamiltonians take the form as portrayed in Table 1.

Table 1: Comparison between the Hamiltonian of Landau levels in Landau gauges and noncommutative oscillator

| Hamiltonian of Landau levels in Landau gauge | Hamiltonian of noncommutative oscillator |
|---|---|
| $\frac{1}{2\mu}\left(\hat{p}_x^2+\hat{p}_y^2\right)+\frac{e^2B^2}{2\mu}\hat{y}^2+\frac{eB}{\mu}\hat{y}\hat{p}_x$ | $\frac{\zeta\theta^2}{4\hbar^2}\left(\hat{p}_x^2+\hat{p}_y^2\right)+\zeta(\hat{x}^2+\hat{y}^2)-\frac{\zeta\theta}{\hbar}\hat{L}_z$ |
| $\frac{1}{2\mu}\left(\hat{p}_x^2+\hat{p}_y^2\right)+\frac{e^2B^2}{2\mu}\hat{x}^2-\frac{eB}{\mu}\hat{x}\hat{p}_y$ | $\frac{\zeta\theta^2}{4\hbar^2}\left(\hat{p}_x^2+\hat{p}_y^2\right)+\zeta(\hat{x}^2+\hat{y}^2)-\frac{\zeta\theta}{\hbar}\hat{L}_z$ |

After solving the time-independent Schrödinger equation, the energy eigenvalues of the noncommutative oscillator in the context of the first Landau gauge are

$$E_{n_y} = \left(2n_y+1\right)\zeta\theta, \tag{33}$$

and the energy eigenstates are

$$\Psi_{n_y}^{(k_0)}(y) = \frac{1}{\sqrt{2^{n_y}n_y!}}\left(\frac{4}{\pi\theta}\right)^{\frac{1}{4}}\exp\left[-\frac{2}{\theta}\left(y+\frac{\theta}{4}k_0\right)^2\right]H_n\left[\frac{4}{\theta}\left(y+\frac{\theta}{4}k_0\right)\right]. \tag{34}$$

Figure 1 and Figure 2 show some of the lower-order states of the isomorphic system and the effect of magnetic field (and hence noncommutativity) on the isomorphic system to provide visual insight.

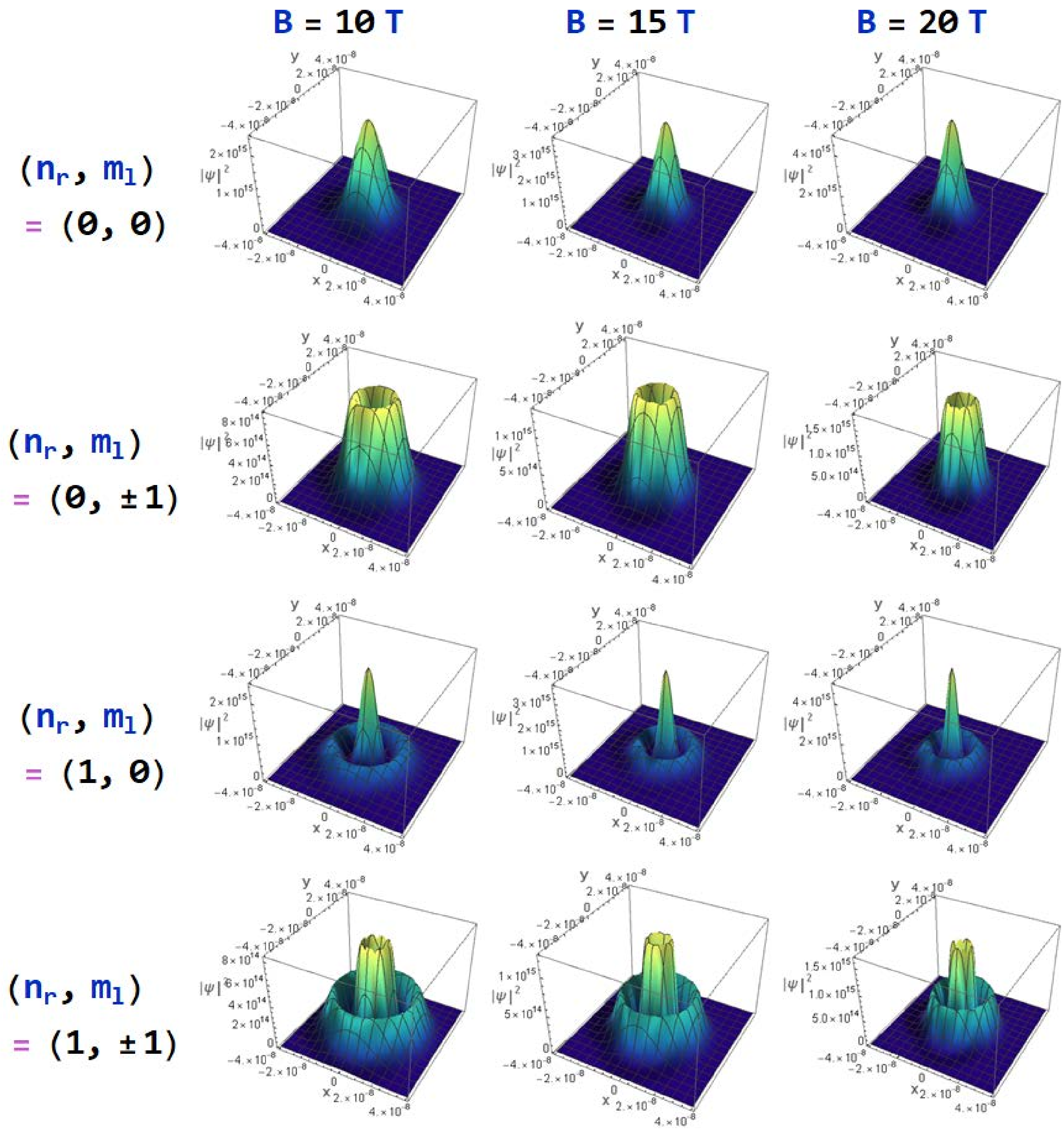


**FIGURE 1.** Some probability density functions of the isomorphic system in symmetric gauge with varying quantum number pairs $(n_r, m_l)$ and magnetic field.

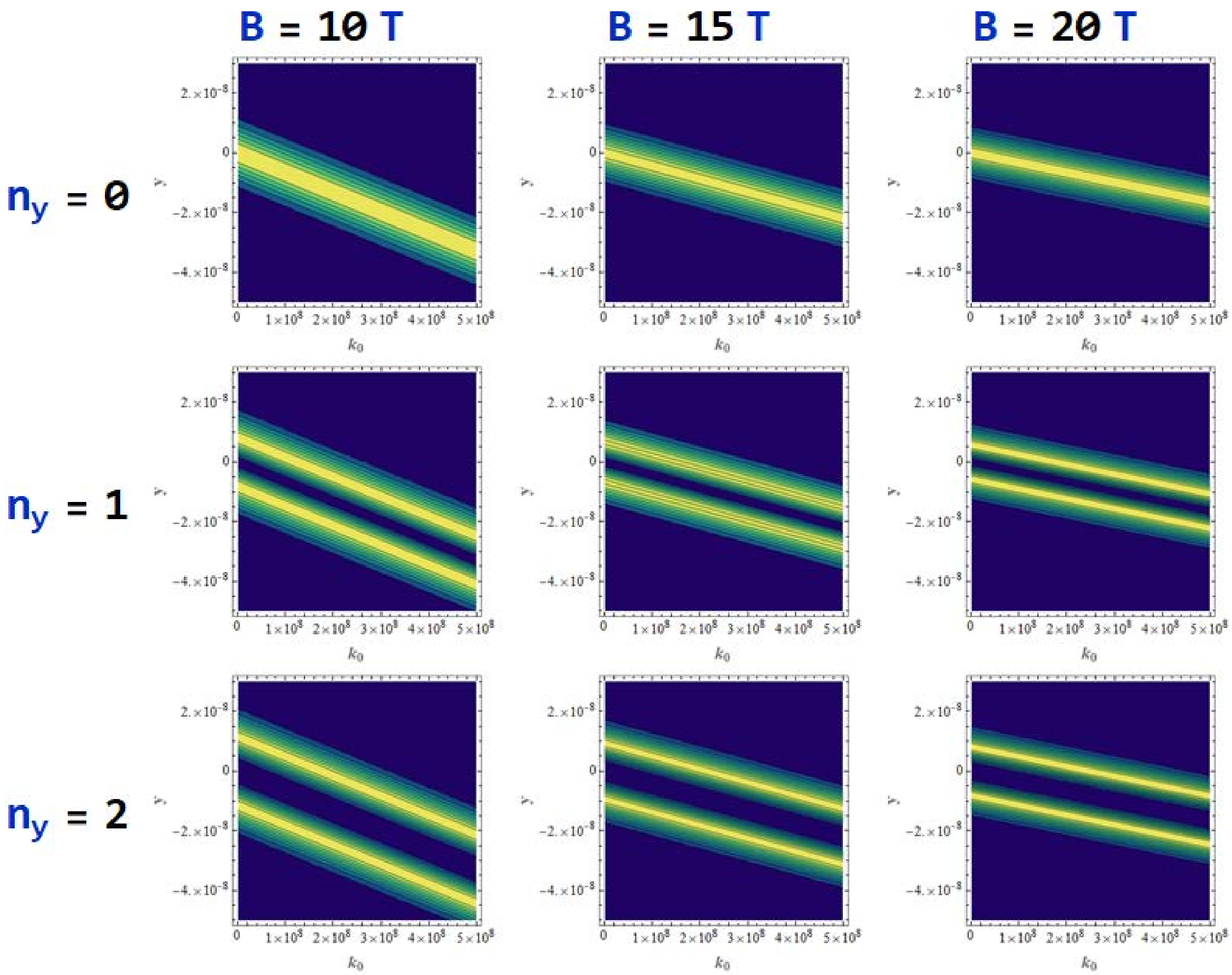


**FIGURE 2.** Some probability density functions of the isomorphic system in Landau gauge with varying quantum number and magnetic field

## IV. CONCLUSION

In this work, we propose an approach to solve noncommutative harmonic oscillator problems within the mathematical formalism of charged particle in magnetic field for the symmetric and Landau gauges. We argue that the equivalence of noncommutative quantum mechanics in the central field and charged particle in magnetic field as portrayed by [12] for the lowest Landau level can be extended to all Landau levels and applied for both gauges in the case of harmonic oscillator. We find that the noncommutative isotropic harmonic oscillator and Landau levels can be considered as an isomorphic system in the context of both Landau and symmetric gauges by satisfying a number of conditions. In the context of symmetric gauge, they are isomorphic up to the similar values of $n_r$ and $m_l$ for $qB = eB > 0$. The noncommutative oscillator also has to be parametrized by a factor $\zeta$ to make the Hamiltonians consistent with each other. As for Landau gauge, the requirements are the same but with an addition that, the noncommutative oscillator has to lose one spatial degree of freedom.

With this research, the problems involving the analytical spectrum of the noncommutative harmonic oscillator can now be understood within the framework of charged particle in magnetic field. This framework provides a simpler and more generalized approach to the noncommutative central field problems since Landau levels are very well understood and it also applies to all energy levels. It is also of major interest to study if this relation still holds when a gauge-invariant transformation with central field potential or any other exactly solvable potential is considered. One can also imagine the possibility of isomorphism if the Hamiltonian is mass, frequency or energy-dependent.

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