Relaxed compact TORI of arbitrary cross sections

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Relaxed (force-free) toroidal plasmas, that is described by the equations $\vec{J}=\vec{\nabla}\times\vec{B}=\mu\vec{B}$, still arises an importance and interest in the field of nuclear fusion. In this article we solve numerically, the relaxed state equation for a CT of arbitrary aspect ratio α and arbitrary geometrical cross sections. The results obtained are in agreement with that of toroids of very large α. Dependence of eigenvalues μ on aspect ratio α is also obtained. Several runs of the program with various wave numbers *k* showed that μ is very insensitive to the choice of *k*. Generally speaking, it is shown that the numerical collocation method used here, works quite well for CT with tight aspect ratio and arbitrary cross section. It gives, with high accuracy, the zero field eigenvalues of the relaxed force-free equation. A good fulfillment of the boundary condition, which describes the relaxed state along the whole boundary for different cross sections, is achieved.

I. INTRODUCTION

It is well known that, the equation

$$\vec{J}=\vec{\nabla}\times\vec{B}=\mu\vec{B}, \tag{1}$$

describes the relaxed toroidal plasmas to a force-free configuration of minimum energy [1,2]. Solution of Eq. (1) arises an importance and interest to describe the gross features of the reversed field pinch (RFP), spheromak configuration, current limitation in toroidal plasmas and others. If μ is spatially uniform then the curl of Eq. (1) gives a vector Helmholtz equation, $\nabla^2\vec{B}+\mu^2\vec{B}=0$ [3]. The divergence of Eq. (1) gives $\vec{B}.\vec{\nabla}\mu=0$ which means that μ is constant on a field line but may vary across field lines (i.e., μ may be thought of as a function of the parameter which labels individual field lines). If there are flux surfaces, then μ is a surface quantity (i.e., μ is constant on a flux surface, but may vary across flux surfaces). Eq. (1) has important limiting cases determined by boundary conditions and also by μ profiles. Two parameters are determining the relaxed state for toroidal system with a perfectly conducting boundary. First, is the magnetic helicity (gauge invariant) $K=\int\vec{A}\bullet\vec{B}\,d\tau$, and second is the toroidal flux Ψ. Also, it is very important to know the parameter μ (the eigenvalue corresponding to relaxed plasma state). In relaxed states, μ cannot exceed the smallest eigenvalue $\mu_{\min}$ and that for a toroidal discharge there is a maximum toroidal current, $I_{\max}$ which is connected to μ. Eigenvalue μ and the normalized field profile are determined by the dimensionless ratio K/Ψ^2 [4]. The value of μ varies continuously with this ratio but it is always below the lowest eigenvalue $\mu_{\min}$ in relaxed state Eq. (1). When the toroidal flux Ψ vanishes, and we have a perefect conducting wall, $\mu_{\min}$ will satisfy the boundary condition:

$$\vec{B}\cdot\vec{n}=0\,. \tag{2}$$

To select the correct minimum energy solution of Eqs. (1) and (2), one can consider the eigenvalues of this system [5]. Solutions of Eqs. (1) and (2) seem to be difficult in toroidal coordinates. Therefore, the force-free magnetic field boundary value problem has been solved in the toroidal coordinates using approximate analytical methods, considering a large, but finite, aspect ratio [6,7]. Toroidal force-free equilibrium has also been considered precisely for tori of rectangluare cross section and finite aspect ratio, and for more general boundaries in the infinite aspect ratio limit [5,7]. Eigenvalues associated with relaxed force-free toroidal plasmas are calculated for tori of arbitrary aspect-ratio and arbitrary cross section (e.g., circle, ellipse, rectangle, …) [8,9]. The results obtained in this case are in agreement with that obtained previously in the limit when the aspect-ratio has a very large value [10].

II. RELAXED STATES SOLUTIONS FOR AXISYMMETRIC CT

To find the solution of Eq. (1) for axisymmetric plasmas, we follow the same methods as per reference [8] , and use cylidrical coordinates (r,φ,z) Fig. 1. In toroidal container, and for finite aspect ratio, the periodicity condition is expressed as $\vec{B}(r,\varphi,z)=\vec{B}(r,\varphi+2\pi,z)$. Assume a seperation of variables as: $F(r,\varphi,z) = f(r)\exp i(m\varphi + kz)$. The meridional cross section in the *r*, *z* plane of the toroidal metallic vessel wall shall be described by an arbitrary curve $z = z(r)$ along which the boundary condition (2) has to be satisfied. Along such an arbitrary curve, and taking into consideration that the field lines are tangetial to this curve, Eq. (2) takes the form:

$$B_r(r,z)\frac{dz}{dr} = B_z(r,z)\,. \qquad (3)$$

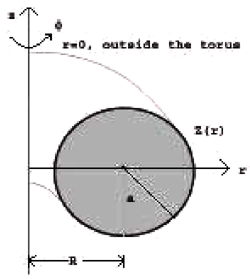

FIG. 1. CT for Relaxed axisymmetric states (torus cross-section).

Later, the curve $z = z(r)$ will be represented by a set of points (collocation points) r_i, z_i which lay on the cross section bounday. From Eq. (1), the following expressions linking the magnetic field components are obtained:

$$\begin{aligned} & i\frac{m}{r}B_z - ikB\varphi = \mu B_r\,, \\ & ikB_r - \frac{\partial B_z}{\partial r} = \mu B\varphi\,, \\ & \frac{1}{r}\frac{\partial (rB_\varphi)}{\partial r} - i\frac{m}{r}B_r = \mu B_z\,. \end{aligned} \qquad (4)$$

For axi-symmetric torus, and inserting the first two equations from Eq. (4) into Eq. (3), we obtain:

$$\begin{aligned} & \frac{\partial (rB_\varphi)}{\partial z}dz + \frac{\partial (rB_\varphi)}{\partial r}dr = d(rB_\varphi) = 0, \\ & \left(\frac{\partial}{\partial \varphi} \to 0,\ m = 0\right) \quad \text{or} \quad rB_\varphi = Const. = C. \end{aligned} \qquad (5)$$

Relation (5) coincide with the zero field condition (2) for $C = 0$. Solution of Eqs. (4) gives the magnetic field components for axisymmetric toroidal plasmas

$$B_r(r,z) = \sum_{l=1}^{N} k_l\ Sin(k_l z)\left[a_l J_1(\mathrm{K}_l r) + b_l Y_1(\mathrm{K}_l r)\right] \qquad (6)$$

$$B_z(r,z) = \sum_{l=1}^{N} \mathrm{K}_l\ Cos(k_l z)\left[a_l J_0(\mathrm{K}_l r) + b_l Y_0(\mathrm{K}_l r)\right] \qquad (7)$$

$$B_\varphi(r,z) = \sum_{l=1}^{N} \mu\ Cos(k_l z)\left[a_l J_1(\mathrm{K}_l r) + b_l Y_1(\mathrm{K}_l r)\right] \qquad (8)$$

where, J_0, J_1 are Bessel functions; Y_0, Y_1 are Neumann functions; a_l, b_l are constant coefficients, and $\mathrm{K}_l = \sqrt{\mu^2 - k_l^2}$. It should be mentioned here that, for the case of a straight cylinder (or for infinite aspect ratio of toroidal plasma), the coefficient b_l goes to zero, since the Neumann functions are singular for $r = 0$. On the otherhand, for finite aspect ratio or CT, as in our case, one has to keep these functions. Inserting Eq. (8) into Eq. (5), the boundary condition for an axisymmetric container of finite aspect ratio and arbitrary cross section reads:

$$\sum_{l=1}^{N} r\mu\ Cos(k_l z)\left[a_l J_1(\mathrm{K}_l r) + b_l Y_1(\mathrm{K}_l r)\right] = C\,. \qquad (9)$$

Eq. (9) describes the magnetic field lines (along which the normal component of the magnetic field vanishes). On a wall made of perfect conductor, the normal component of $\vec{B}$ vanishes too. When the constant $C = 0$, Eq. (9) will coincide with the zero field condition.

III. EIGENVALUES $(\mu's)$ FOR CT WITH ARBITRARY CROSS SECTION

Let us consider a CT with circular cross section. In this case, Z in Eq. (9) could be represented by:

$$z_i(r) = \sqrt{a^2 - (r_i - R)^2}, \quad i = 1,2,\ldots,P\,. \qquad (10)$$

Since rB_φ is symmetric in Z but not in *r*, the collocation points have been distributed over the whole upper half of the cross section. For infinite aspect ratio, the boundary condition reads: $B_\varphi = a_1 J_0(\mu a) = 0$, which yields eigenvalue $\mu a = 2.4048$ [5]. Table I, gives the lowest eigen values, for CT of circular cross section and aspect ratios $\alpha = 2, 1.5$ ($R = 2, 1.5$ and $a = 1$). When increasing the aspect ratio to values of $\alpha = 10, 20$ (cylinderical approximation), the zero field eigenvalue μa decreases and tends to the value 2.4048 [8] for axisymetric $m = 0$ mode, which obtained by Taylor [2,5] for infinite aspect ratio. For circular cross section, μ increases with decrease of α. One might argue that the choice of the wave numbers k_l is highly arbitrary and influence the result for μ. Therefore, for several runs for the program with various k_l it is found that μ is very insensitive to the choice of k_l as indicated in Table I.

All the obtained results have been calculated for a circle represented by the collocation points derived from relation (10). This circle, along which the boundary condition is fulfilled, must also be described by Eq. (10) for all points lay on it. Therefore, we plotted the function given by Eq. (9) for $C=0$; $C \neq 0$. The result is shown in Fig. 2 for *R* = 1.5, *a* = 1, *N* = 6 modes (k_l = 0.001, 0.4, 0.6, 1.7, 1.9, 2.0) and eigenvalue $\mu a = 2.47372813$. The collocation points r_i, z_i are marked by small squares and it is clear that it is well fitted on the boundary. For $C=0$, this gives the poloidal magnetic field line along the container, and when *C* has different values, this gives the magnetic field lines inside the container (numbers written on the field lines are the different values of *C*) **.** The outward toroidal shift is clearly seen for small aspect ration. Above procedure is also performed for other cross sections. A multipinch, is described here by a Cassini curve shifted by the torus major radius $R:(u^2+z^2)^2+2b^2(u^2-z^2)^2-a^4+b^4=0$, $r-R=u$. Cassini curve is identical to a multi-pinch configuration if $b<a<b\sqrt{2}$ with half-width given by $\Delta^2=a^2-b^2$. Fig. 3 shows a Cassini curve container with: eigenvalue μ = 3.2133226, *R* = 1, *a* = 0.35, *b* = 0.3, *N* = 6 modes (k_l = 0.01, 0.1, 1.0, 1.3, 1.8, 2.1). When *R* = 2, *a* = 1, *b* = 0.5, the corresponding eigenvalue becomes μ = 2.50059 [8]. A D-shaped cross section is also investigated in Fig. 4, with an eigenvalue μ = 1.5050883. A trial is also made to apply above methods to a sharp edged cross section (e.g., astroidal-like shape). Results are given in Table II, and Fig. 5. It is clear that magnetic field lines are smoothly connected at vortices.

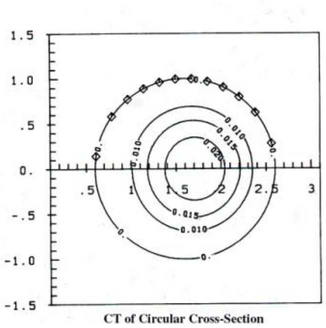

FIG. 2. CT of Circular cross-section.

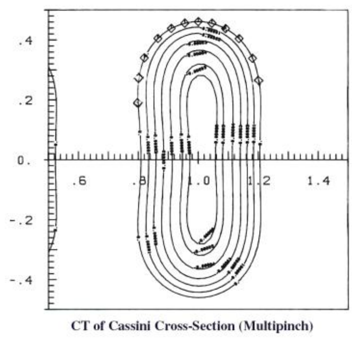

FIG. 3. CT of Cassini cross-section (Multipinch) .

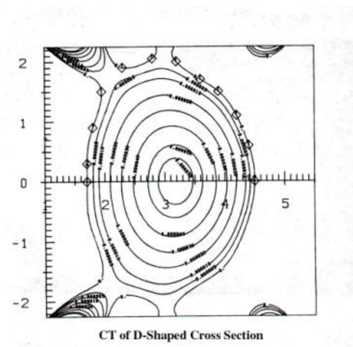

FIG. 4. CT of D-Shaped cross-section.

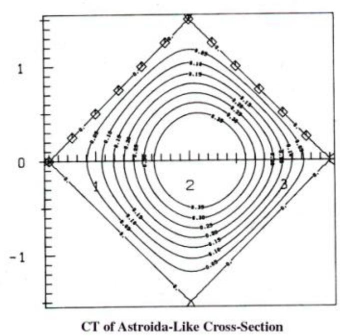

FIG. 5. CT of Astroidal-Like cross-section.

TABLE I. Dependence of μ on the choice of k_l and from aspect ratio

k_1	k_2	k_3	k_4	k_5	k_6	$\alpha = 2$ μ	$\alpha = 1.5$ μ
0.001	0.4	0.6	1.7	1.9	2.0	2.44692724	2.47372813
0.01	0.1	1.0	1.5	2.0	2.1	2.44692709	2.47372830
0.1	0.5	0.6	1.3	1.8	2.0	2.44692762	2.47372850
1.0	1.3	1.5	1.7	1.9	2.1	2.44692696	2.47372901

TABLE II. Eigenvalue decreases with increasing the elongation

k_1	k_2	k_3	k_4	k_5	k_6	$R = 2; a = b = 1.5$ μ	$R = 2; a = 1; b = 1.5$ μ
0.001	0.3	0.5	1.6	1.7	1.8	3.21633644	2.66830312
0.01	0.1	1.1	1.4	1.6	1.7	3.21633240	2.66831130
0.1	0.5	0.6	1.3	1.5	1.7	3.21633541	2.66830214
1.0	1.3	1.5	1.6	1.7	1.9	3.21633512	2.66830453

IV. CONCLUSION

In conclusion**,** it is shown that the numerical method (collocation method) [8,11] works quite well for CT with tight aspect ratio and arbitrary cross section. It gives, with high accuracy, the zero field eigenvalues of the relaxed force-free equation. A good fulfilment of the boundary condition, which described the relaxed state along the whole boundary for different cross sections, is achieved.

As application to the problem investigated here, the solar prominences [3,12] correspond to solutions of Eq. (1) with non-axisymmetric boundary conditions and, instead of having a finite volume, one boundary is a planar surface and the other boundaries are at infinity. Boundary conditions are imposed on the planar surface. μ is usually non-uniform since it is obviously energetically impossible to fill up a semi-infinite space with uniform μ. Shapes of cross sections and aspect ratios are found to affect the eignenvalues of relaxed plasma state, hence they influence strongly on the physical picture of toroidal plasmas. It would be also of interest to check our methods for the cases when $\alpha \to 1$, coupling magnetic and fluid aspects of plasma (i.e., for $\vec{J} \times \vec{B} \cong \vec{\nabla} P \neq 0$) and for nonaxisymetric case, with application to space physics.

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