Extending Bianchi type-I and type-II homogeneous three manifolds to four dimensions

Aslam Abdullah[a,c], Hishamuddin Zainuddin[a,b] and Zainul Abidin Hassan[b]
[a]Laboratory of Computational Sciences & Informatics, Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia
[b]Physics Department, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia
[c]Jabatan Kejuruteraan Loji dan Automotif, Fakulti Kejuruteraan Mekanikal dan Pembuatan, Universiti Tun Hussein Onn Malaysia, 86400 Parit Raja, Batu Pahat, Johor Darul Takzim

(Received 7 April 2007)

Type-I and Type-II homogeneous cosmologies described by three-parameter isometry groups G_3 with transitive actions on three-manifolds were extended to those involving transitive groups of isometries G_4 with an additional dimension of four-manifolds. In order to extend a three-manifold metric which admits the Bianchi type-N (N = I, II) to that of four-manifold, the group G_4 was specified via a translational symmetry over the extra dimension. The extended metric of types I & II are solved explicitly.

I. INTRODUCTION

One of the questions of interest in cosmological study is related to the dynamics of the universe. The acceleration behaviour of the universe becomes a *popular* topic among the cosmologists, since it is what recent observational data indicate rather unexpectedly [1]. In order to propose the cosmological models which exhibit this behaviour, an avenue would be introducing extra dimensions. These extra dimensions of course, are not considered without reason, since their possibility has already been proposed by both the string and the Kaluza-Klein theories.

Usually, the brane worlds are considered to be of five-dimensions [2], where there is only one extra dimension along which the branes are positioned. Both the bulk and the branes *share* one time-dimension.

In this note, the Killing equation solutions for four-manifold calculated also involves single extra dimension. The three-dimensional manifolds embedded in those of four-dimensions are spacelike sections of the branes. Thus, the manifolds of four dimensions are the spacelike sections of the bulk, proposed in this note. By considering that these bulks are homogeneous, they must admit a transitive group of motions. This constraint ties-up spacelike sections of the brane worlds with the concept of symmetry.

II. THE KILLING EQUATIONS

The Killing equations describing symmetries of manifold with metric *g* are given by

$$\sum\{\xi^{\alpha}\partial_{\alpha}g_{\mu\nu} + g_{\mu\alpha}\partial_{\nu}\xi^{\alpha} + g_{\nu\alpha}\partial_{\mu}\xi^{\alpha}\} = 0\,, \qquad (1)$$

where ξ^{α} and $g_{\mu\nu}$ are the components of the Killing vector *X* and the manifold metric, respectively. For any three-dimensional metric manifold, the Killing vector *X* is given by

$$X = \xi^2\partial_2 + \xi^3\partial_3 + \xi^4\partial_4\,.$$

In order to find the Type-I and Type-II four-manifolds incorporating such symmetries i.e. $\mathbf{B}_I^4$ and $\mathbf{B}_{II}^4$, we have to solve the Killing equations.

Setting $\alpha = 2,3,4$, Eq. (1) becomes

$$\sum_{\alpha=2}^{4}\{\xi^{\alpha}\partial_{\alpha}g_{\mu\nu} + g_{\mu\alpha}\partial_{\nu}\xi^{\alpha} + g_{\nu\alpha}\partial_{\mu}\xi^{\alpha}\} = 0\,, \qquad (2)$$

where $(\mu,\nu) = (1,1)$, (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4). Note that $g_{\mu\nu}$ in this system are the components of the four-manifold metric.

The maximum number of the group parameters [3] describing symmetries of the three-manifold is given by

$$r = \frac{n(n+1)}{2} = 6,$$

since $n = 3$ is the dimension of the manifold. It corresponds to the maximum number of the Killing vectors which generate the complete group of isometries for three-manifold. For every combination of (μ,ν) then, there are six different sets of the Killing vector's components ξ^2, ξ^3, ξ^4. Thus Eq. (2) consists of sixty equations, in general.

The components of the Killing vectors (ξ^2, ξ^3, ξ^4) which generate the Type-N are given as follows[1]:

Type-I (1,0,0), (0,1,0), (0,0,1).

Type-II (0,1,0), (0,0,1), $(-1, x^4, 0)$.

Since the Killing vectors are only three, the number of non-trivial equations in Eq. (2) is reduced to thirty for each case. Substituting these components into Eq. (2), we can find the solutions as shown in the following sections.

Assuming $\xi^1 \neq 0$, so that the complete group of isometries for a four-manifold $\mathbf{B}^4$ is generated by

$$X = \xi^1 \partial_1 + \xi^2 \partial_2 + \xi^3 \partial_3 + \xi^4 \partial_4, \quad (3)$$

Eq. (1) becomes

$$\sum_{\alpha=1}^{4} \{\xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha} \partial_\nu \xi^\alpha + g_{\nu\alpha} \partial_\mu \xi^\alpha\} = 0. \quad (4)$$

Note that if ξ^1 is taken to be zero, naturally there are only three Killing vectors generating the Type-N.

III. FOUR-DIMENSIONAL TYPE-I MANIFOLD

All components of the Type-I four-manifold's metric depend only on the extra variable x^1 and we set $g_{11} = 1$ The line element for $\mathbf{B}_I^4$ is then represented in geodetic form as

$$ds^2 = dx^{1^2} + \sum_{\mu=1,\nu,\kappa=2}^{4} (g_{\mu\nu} dx^\mu dx^\nu + g_{1\kappa} dx^1 dx^\kappa), \quad (5)$$

where $g_{\mu\nu}$ being functions only of x^1, $(\mu,\nu) \neq (1,1)$ such that

$$g_{12} = \phi^2(x^1),\ g = \psi^2(x^1),\ g_{14} = \gamma^2(x^1),\ g_{22} = t^2(x^1),$$
$$g_{23} = \eta^2(x^1),\ g_{24} = \lambda^2(x^1),\ g_{33} = \delta^2(x^1),$$
$$g_{34} = \alpha^2(x^1),\ g_{44} = \beta^2(x^1).$$

We assume that the resultant four-manifold metric is invariant under the translations along the extra coordinate lines x^1 so that the Killing vectors Lie algebra formed by

$$X_1 = \partial_2,\ X_2 = \partial_3,\ X_3 = \partial_4,\ X_4 = \partial_1.$$

The components of the Killing vector Eq. (3) are then given by

$$\xi^1 = \xi^2 = \xi^3 = \xi^4 = 1.$$

Substituting these components into Eq. (4) gives

$$\phi^2 = a,\ \psi^2 = b,\ \gamma^2 = c,\ t^2 = d,\ \eta^2 = e,$$
$$\lambda^2 = f,\ \delta^2 = g,\ \alpha^2 = h,\ \beta^2 = i,$$

where *a, b, c, d, e, f, g, h, i* are arbitrary constants.

The line element Eq. (5) is then given by

$$ds^2 = dx^{1^2} + \sum_{\mu=1,\nu,\kappa=2}^{4} (g_{\mu\nu} dx^\mu dx^\nu + g_{1\kappa} dx^1 dx^\kappa),$$

where

$$g_{12} = a,\ g_{13} = b,\ g_{14} = c,\ g_{22} = d,\ g_{23} = e,$$
$$g_{24} = f,\ g_{33} = g,\ g_{34} = h,\ g_{ii} = i,$$

Since all the coefficients of the differential form are constants we have

$$ds^2 = dx^{1^2} + dx^{2^2} + dx^{3^2} + dx^{4^2},$$

with a (linear) change of variables. The resultant extended manifold then has zero curvature (i.e. flat manifold) as expected.

IV. FOUR-DIMENSIONAL TYPE-II MANIFOLD

We work out the extension in a similar way for Type-II. The metric components are taken to be

$$g_{11} = 1,\ g_{12} = \phi^2(x^1),\ g_{13} = \psi^2(x^1),\ g_{22} = \theta^2(x^1),$$
$$g_{23} = \chi^2(x^1),\ g_{33} = \zeta^2(x^1),$$
$$g_{14} = \psi^2(x^1)x^2 + \varphi^2(x^1),$$
$$g_{24} = \chi^2(x^1)x^2 + \varsigma^2(x^1),$$
$$g_{34} = \zeta^2(x^1)x^2 + \gamma^2(x^1),$$
$$g_{44} = \zeta^2(x^1)x^{2^2} + 2\gamma^2(x^1)x^2 + \beta^2(x^1),$$

[1]Note that number of all types is nine, but seven of them (N=III, IV, V, VI, VII, VIII and IX) are not worked out here.

where $\phi^2(x^1)$, $\psi^2(x^1)$, $\theta^2(x^1)$, $\chi^2(x^1)$, $\zeta^2(x^1)$, $\varphi^2(x^1)$, $\varsigma^2(x^1)$, $\gamma^2(x^1)$ and $\beta^2(x^1)$ are functions of x^1.

Assuming the translational Killing symmetry over the extra variable x^1, the Killing vectors become

$$X_1 = \partial_3,$$
$$X_2 = \partial_4,$$
$$X_3 = -\partial_2 + x^4\partial_3,$$
$$X_4 = \partial_1.$$

Thus, the components of the Killing vector Eq. (3) are given by

$$\xi^1 = 1,$$
$$\xi^2 = -1,$$
$$\xi^3 = 1 + x^4,$$
$$\xi^4 = 1.$$

Substituting these components into Eq. (4) we have

$$g_{11} = 1,\ g_{12} = a,\ g_{13} = b,\ g_{22} = d,\ g_{23} = e,\ g_{33} = g,$$
$$g_{14} = bx^2 + c,$$
$$g_{24} = ex^2 + f,$$
$$g_{34} = gx^2 + h,$$
$$g_{44} = gx^{2^2} + 2hx^2 + i,$$

where *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h* and *i* are arbitrary constants.

The line element can now be given as

$$ds^2 = dx^{1^2} + adx^1dx^2 + bdx^1dx^3 + (bx^2 + c)dx^1dx^4$$
$$+ ddx^{2^2} + edx^2dx^3 + (ex^2 + f)dx^2dx^4$$
$$+ gdx^{3^2} + (gx^2 + h)dx^3dx^4 + (gx^{2^2} + 2hx^2 + i)dx^{4^2}.$$

V. CONCLUSION

Both four-dimensional manifolds $\mathbf{B}_I^4$ and $\mathbf{B}_{II}^4$ still (re-)admit the Bianchi type-I and II, respectively. Their metric components satisfy the Killing equations involving the components of the Killing vector *X* which generates the complete group of isometries for the corresponding three-manifolds. It is obvious that the (incomplete) group of isometries for the Type-N (N = I, II) three-manifolds is a subgroup of that for the resulted four-manifolds. This allows us to assume the additional translation Killing symmetry over the extra variable x^1 as shown in all two cases.

ACKNOWLEDGEMENT

The authors are very grateful to the Institute of Advanced Technology at Universiti Putra Malaysia, where this work was initiated. This work was financially supported by the Ministry of Science, Technology and Innovation under IRPA grant (Project no.: 09-02-04-284 EA001).

REFERENCES

[1] C. M. Chen, P. M. Ho, I. P. Neupane, N. Ohta, and J. E. Wang, "Hyperbolic Space Cosmologies", JHEP, **0310**, 058 (2003).
[2] P. Brax, and C.v.d. Bruck, "Cosmology and Brane Worlds: A review", Class. Quant. Grav., **20**, R201-R232 (2003).
[3] L. Bianchi, "On the Three-Dimensional Manifolds which Admit a Continuous Group of Motions" (Eng. Trans.), *Memorie di Matematica e di Fisica della Sicieta Italiana delle Scienze Serie Terza,* Tomo, **XI**, 267-352 (1898).