# A numerical investigation on the MAP solutions of the SU(2) YMH theory

K. M. Wong and Rosy Teh
*School of Physics, Universiti Sains Malaysia, 11800 USM, Penang*

(Received 6 December 2006)

We notice that it is not necessary to describe the monopole-antimonopole pair (MAP) and monopole-antimonopole chains (MAC) solutions in terms of $\theta$-winding number *m* greater than one as they can also be parameterized by a single integer *s*. Here we study the MAP solution of the SU(2) Yang-Mills-Higgs theory which belongs to the topological trivial sector. This solution is parameterized by $\theta$-winding number one, $\phi$-winding number one and integer *s* = 1.

## I. INTRODUCTION

The SU(2) Yang-Mills-Higgs (YMH) field theory, with the Higgs field in the adjoint representation, are known to possess both the magnetic monopole [1,2] and multimonopole solutions [3]. Monopole solutions with unit magnetic charge and finite energy are spherically symmetric [1,2]. Multimonopole with finite energy cannot be spherically symmetric [4] and possess at most axial symmetry [3].

In the limit of vanishing Higgs potential, the Bogomol'nyi-Prasad-Sommerfield (BPS) limit, exact monopole and multimonopole solutions [2,3] are known. These BPS solutions possess finite and minimal energies. However, when the Higgs potential is finite, only numerical solutions [1] are known.

Recently Kleihaus *et al*. constructed non-Bogomol'nyi BPS solutions which satisfy only the second order field equations but not the first order Bogomol'nyi equations. Their solutions possess only axial symmetry and correspond to a monopole-antimonopole pair (MAP) [5], and monopole-antimonopole chain (MAC) [6]. These MAP and MAC solutions are parameterized by $\theta$-winding number *m* (> 1) and $\phi$-winding number *n* = 1.

In Ref. [7], we show that the $\theta$-winding number of the MAP and MAC solutions can be reduced to one and a single integer parameter *s*. In other words, there exist an equivalent form of the solutions with normal $\theta$-winding number *m* = 1 and integer parameter *s* for all the solutions of $\theta$-winding number *m*.

In this paper we compute the numerical MAP solutions with *m* = 1, *s* = 1 and *m* = 2, *s* = 0. Both the asymptotic conditions *s* = 0 and *s* = 1 correspond to a pure gauge at spatial infinity and hence the system possess zero net magnetic charge. We connect the asymptotic solutions with the trivial vacuum at the origin numerically by using mathematical softwares Maple and Matlab.

We briefly review the SU(2) YMH field theory in the next section. We present the magnetic ansatz and some of its basic properties in section III. In section IV, we discussed the net magnetic charge of the configuration and we present numerical results in section V. We conclude our results and give some comments in section VI.

## II. THE SU(2) YANG-MILLS-HIGGS-THEORY

The SU(2) YMH theory admits the triplet gauge field $A^a_\mu$ which are the Yang-Mills vector fields coupled to a scalar Higgs triplets field $\Phi^a$ in 3+1 dimensions. The index *a* is the SU(2) internal space index and for a given *a*, $\Phi^a$ is a scalar whereas $A^a_\mu$ is a vector under Lorentz transformation. The Lagrangian in 3+1 dimensions is given by

$$L = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - \frac{1}{4}\lambda\left(\Phi^a\Phi^a - \frac{\mu^2}{\lambda}\right)^2, \tag{1}$$

where the Higgs field mass, $\mu$, and the strength of the Higgs potential, $\lambda$, are constants. The vacuum expectation value of the Higgs field is then given by $\mu/\sqrt{\lambda}$. The Lagrangian (1) is gauge invariant under the set of independent local SU(2) transformations at each space time point. The covariant derivative of the Higgs field and the gauge field strength tensor are given respectively by

$$D_\mu\Phi^a = \partial_\mu\Phi^a + \epsilon^{abc}\, A^b_\mu\Phi^c \text{, and}$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}\, A^b_\mu A^c_\nu. \tag{2}$$

Since the gauge field coupling constant *g* can be scaled away, we can set *g* to one without any loss of generality.

The metric used is $g_{\mu\nu} = (-+++)$. The SU(2) internal group indices $a, b, c$ run from 1 to 3 and the spatial indices are $\mu$, $\nu$, $\alpha = 0, 1, 2$, and 3 in Minkowski space.

The equations of motion that follow from the Lagrangian (1) are

$$D^\mu F^a_{\mu\nu} = \partial^\mu F^a_{\mu\nu} + \in^{abc} A^{b\mu} F^c_{\mu\nu} = \in^{abc} \Phi^b D_\nu \Phi^c,$$
$$D^\mu D_\mu \Phi^a = -\lambda \Phi^a \left( \Phi^b \Phi^b - \frac{\mu^2}{\lambda} \right). \quad (3)$$

The 't Hooft electromagnetic field tensor as proposed by 't Hooft [1], is given by

$$F_{\mu\nu} = \hat{\Phi}^a F^a_{\mu\nu} - \in^{abc} \hat{\Phi}^a D_\mu \hat{\Phi}^b D_\nu \hat{\Phi}^c$$
$$= \partial_\mu A_\nu - \partial_\nu A_\mu - \in^{abc} \hat{\Phi}^a \partial_\mu \hat{\Phi}^b \partial_\nu \hat{\Phi}^c, \quad (4)$$

where $A_\mu = \hat{\Phi}^a A^a_\mu$, the Higgs field unit vector $\hat{\Phi}^a = \Phi^a / |\Phi|$ and the Higgs field magnitude $|\Phi| = \sqrt{\Phi^a \Phi^a}$. The Abelian electric field is $E_i = F_{0i}$, and the Abelian magnetic field is $B_i = -\frac{1}{2} \in_{ijk} F_{jk}$. The topological magnetic current [8] which is also the topological current density of the system is defined as

$$k_\mu = \frac{1}{8\pi} \in_{\mu\nu\rho\sigma} \in_{abc} \partial^\nu \hat{\Phi}^a \partial^\rho \hat{\Phi}^b \partial^\sigma \hat{\Phi}^c, \quad (5)$$

and the corresponding conserved topological magnetic charge is

$$M = \int d^3x k_0 = \frac{1}{8\pi} \int \in_{ijk} \in^{abc} \partial_i (\hat{\Phi}^a \partial_j \hat{\Phi}^b \partial_k \hat{\Phi}^c) d^3x$$
$$= \frac{1}{8\pi} \oint d^2\sigma_i (\in_{ijk} \in^{abc} \hat{\Phi}^a \partial_j \hat{\Phi}^b \partial_k \hat{\Phi}^c) \quad (6)$$
$$= \frac{1}{4\pi} \oint d^2\sigma_i B_i.$$

Our work is restricted to the static case where $A^a_0 = 0$. Hence the conserved energy of the system for the static case reduces to

$$E = \int d^3x \left( \frac{1}{2} B^a_i B^a_i + \frac{1}{2} D_i \Phi^a D_i \Phi^a + \frac{1}{4} \lambda \left( \Phi^a \Phi^a - \frac{\mu^2}{\lambda} \right)^2 \right). \quad (7)$$

Here $i, j, k$ which are the three space indices run from 1, 2, and 3. This energy vanishes when the gauge potential, $A^a_i$ is zero or when $A^a_i$ is a pure gauge, and when $\Phi^a \Phi_a = \mu^2 / \lambda$ and $D_i \Phi^a = 0$. In this paper, we consider the case with vanishing Higgs potential, $\lambda = 0$.

## III. THE MAGNETIC ANSATZ

We make use of the static magnetic ansatz [5,6] to solve for the monopoles solutions here. The gauge fields and the Higgs field are given respectively by

$$A^a_i = \frac{1}{r}(-\psi_1 \hat{\phi}^a \hat{\theta}_i + \psi_2 \hat{\theta}^a \hat{\phi}_i) + \frac{1}{r}(R_1 \hat{\phi}^a \hat{r}_i - R_2 \hat{r}^a \hat{\phi}_i),$$
$$\Phi^a = \Phi_1 \hat{r}^a + \Phi_2 \hat{\theta}^a, \quad (8)$$

where $\psi_1$, $\psi_2$, $R_1$, $R_2$, $\Phi_1$ and $\Phi_2$ are profile functions of $r$ and $\theta$. The spherical coordinate orthonormal unit vectors, $\hat{r}_i$, $\hat{\theta}_i$, and $\hat{\phi}_i$ are defined by

$$\hat{r}_i = \sin\theta \cos\phi \, \delta_{1i} + \sin\theta \sin\phi \, \delta_{2i} + \cos\theta \, \delta_{3i},$$
$$\hat{\theta}_i = \cos\theta \cos\phi \, \delta_{1i} + \cos\theta \sin\phi \, \delta_{2i} - \sin\theta \, \delta_{3i},$$
$$\hat{\phi}_i = -\sin\phi \, \delta_{1i} + \cos\phi \, \delta_{2i}, \quad (9)$$

and the isospin coordinate orthonormal unit vectors, $\hat{r}^a$, $\hat{\theta}^a$, and $\hat{\phi}^a$ are defined by

$$\hat{r}^a = \sin m\theta \cos n\phi \, \delta^a_1 + \sin m\theta \sin n\phi \, \delta^a_2 + \cos m\theta \, \delta^a_3,$$
$$\hat{\theta}^a = \cos m\theta \cos n\phi \, \delta^a_1 + \cos m\theta \sin n\phi \, \delta^a_2 - \sin m\theta \, \delta^a_3,$$
$$\hat{\phi}^a = -\sin n\phi \, \delta^a_1 + \cos n\phi \, \delta^a_2, \quad (10)$$

where $r = \sqrt{x^i x_i}$, $\theta = \cos^{-1}(x_3 / r)$, and $\phi = \tan^{-1}(x_2 / x_1)$.

With ansatz (8) the field equations (3) reduce to six PDEs in $r$ and $\theta$. The ansatz possesses a residual U(1) gauge symmetry. To fix the gauge, we impose the gauge condition $r\partial_r R_1 - \partial_\theta \psi_1 = 0$.

In Refs. [5,6], Kleihaus *et al.* consider MAP and MAC solutions which is parameterized by the $\theta$-winding number $m$ (> 1) and $\phi$-winding number $n$ (= 1). In Ref [7], we show that there always exist an equivalent form of the solutions with normal winding number $m$ = 1 and an integer $s$ for all the solutions with $\theta$ winding number.

The MAP solution possesses exact asymptotic solutions at both small and large $r$. Upon reducing these solutions to the $m$ = 1 form with integer $s$, both the MAP and MAC solutions correspond to the trivial vacuum ($s$ = 0) at small $r$. However, at large $r$, the MAP solutions tend to a different sector of the vacuum with parameter $s$ = 1, 2, 3… Hence the MAP solutions correspond to a one monopole-antimonopole pair when $s$ = 1, a two monopole-antimonopole pair when $s$ = 2, and so on with net topological magnetic charge zero. Here we compute numerically the MAP solution with $m$ = 1, $s$ = 1, $n$ = 1 and $m$ = 2, $s$ = 0, $n$ = 1.

The boundary conditions at the origin are

$$\psi_1(0,\theta)\to 0\,,\ \psi_2(0,\theta)\to 0\,,$$
$$R_1(0,\theta)\to 0\,,\ R_2(0,\theta)\to 0\,,$$
$$\Phi_1(0,\theta)\to -\xi\cos\theta\,,\ \Phi_2(0,\theta)\to \xi\sin\theta\,, \quad (11)$$

where $\xi$ is arbitrary constant and the boundary conditions at $r$ infinity are

$$\psi_1(\infty,\theta)\to 2\,,\ \psi_2(\infty,\theta)\to 2n\,,$$
$$R_1(\infty,\theta)\to 0\,,\ R_2(\infty,\theta)\to 0\,,$$
$$\Phi_1(\infty,\theta)\to \cos\theta\,,\ \Phi_2(\infty,\theta)\to \sin\theta\,. \quad (12)$$

Regularity on the $z$-axis (at $\theta=0$ and $\theta=\pi$) requires

$$R_1=0\,,\ R_2=0\,,\ \Phi_2=0\,,$$
$$\partial_\theta\psi_1=0\,,\ \partial_\theta\psi_2=0\,,\ \partial_\theta\Phi_1=0\,. \quad (13)$$

The equations of motion (3) are then solved numerically by using ansatz (8) with the boundary conditions (11)-(13). The constant $\xi$ will be determined by the results of the numerical calculations.

## IV. THE NET MAGNETIC CHARGE

To calculate for the Abelian magnetic field $B_i$, we rewrite the Higgs field in Eq. (8) from the spherical to the Cartesian coordinate system, [5,6]

$$\Phi^a=\Phi_1\hat{r}^a+\Phi_2\hat{\theta}^a+\Phi_3\hat{\phi}^a=\tilde{\Phi}_1\delta^{a1}+\tilde{\Phi}_2\delta^{a2}+\tilde{\Phi}_3\delta^{a3}, \quad (14)$$

where

$$\tilde{\Phi}_1=\sin m\theta\cos n\phi\,\Phi_1+\cos m\theta\cos n\phi\,\Phi_2-\sin n\phi\,\Phi_3$$
$$=|\Phi|\cos\alpha\sin\beta,$$
$$\tilde{\Phi}_2=\sin m\theta\sin n\phi\,\Phi_1+\cos m\theta\sin n\phi\,\Phi_2+\cos n\phi\,\Phi_3$$
$$=|\Phi|\cos\alpha\cos\beta,$$
$$\tilde{\Phi}_3=\cos m\theta\,\Phi_1-\sin m\theta\,\Phi_2=|\Phi|\sin\alpha\,. \quad (15)$$

The Higgs unit vector is then simplified to

$$\hat{\Phi}^a=\cos\alpha\sin\beta\,\delta^{a1}+\cos\alpha\cos\beta\,\delta^{a2}+\sin\alpha\,\delta^{a3}\,, \quad (16)$$

where

$$\sin\alpha=\frac{\Phi_1\cos m\theta-\Phi_2\sin m\theta}{\sqrt{\Phi_1^2+\Phi_2^2}}\,,$$
$$\beta=\frac{\pi}{2}-\phi\,, \quad (17)$$

and the Abelian magnetic field is found to be

$$B_i=-\frac{1}{r^2\sin\theta}\left(\frac{\partial\sin\alpha}{\partial\theta}\frac{\partial\beta}{\partial\phi}-\frac{\partial\sin\alpha}{\partial\phi}\frac{\partial\beta}{\partial\theta}\right)\hat{r}_i$$
$$+\frac{1}{r\sin\theta}\left(\frac{\partial\sin\alpha}{\partial\phi}\frac{\partial\beta}{\partial r}-\frac{\partial\sin\alpha}{\partial r}\frac{\partial\beta}{\partial\phi}\right)\hat{\theta}_i\,. \quad (18)$$

The magnetic charge is defined as

$$4\pi M=\oint d^2\sigma_i B_i=\int B_i(r^2\sin\theta\, d\theta)\hat{r}_i\, d\phi\,. \quad (19)$$

Here we define the magnetic charge enclosed by the upper hemisphere of infinite radius as $M_+$, whereas the magnetic charge enclosed by the lower hemisphere of infinite radius is denoted by $M_-$. The value of $M_+$ is calculated to be

$$M_+=-\frac{1}{2}\sin\alpha\Big|_{0,r\to\infty}^{\pi/2}=+1\,. \quad (20)$$

The upper hemisphere then possesses a positive charged monopole. Similarly, considering the lower hemisphere by integrating Eq. (19) from $\theta=\pi/2$ to $\theta=\pi$ gives

$$M_-=-\frac{1}{2}\sin\alpha\Big|_{\pi/2,r\to\infty}^{\pi}=-1\,, \quad (21)$$

which correspond to a negative charged antimonopole.

These calculations show indeed that the configuration possesses a monopole-antimonopole pair, with the monopole situated on the positive $z$-axis and the antimonopole at equidistance on the negative $z$-axis. At $r$ infinity for surface enclosing both charges, their contributions compensate and yields zero net magnetic charge.

## V. THE NUMERICAL RESULTS

The asymptotic conditions (12) correspond to a pure gauge vacuum at spatial infinity. We connect this asymptotic condition with the trivial vacuum at the origin (11) numerically by using mathematical softwares Maple and Matlab.

The equations of motion are discretized on a non-equidistant grid, covering the integration region $0\le X\le 1$ and $0\le\theta\le\pi$, where $X$ is the finite interval compactified coordinate. The $X$ and $\theta$ grid are subdivided into $M$ and $N$ divisions. The best accuracy our computer is able to support is $M = 40$ and $N = 25$.

The numerical method used is the Gauss-Newton method and it is a good iterative method to obtain numerically accurate solutions. First the set of six PDEs are transformed into a system of non-linear equations by considering second order finite difference approximation. After providing good initial guess to the system of non-linear equations, the solver will iterate and converge to the true numerical answers.

Our result confirms that the boundary conditions (11)-(13) corresponds to monopole-antimonopole pair solution. In Fig. (1) we plot the modulus of the Higgs field as a functions of $\rho$ and $z$ where $\rho = \sqrt{x^2 + y^2}$. The magnetic poles are located at the points where the modulus of Higgs field is zero. In particular, when $M = 40$, $N = 25$, the location where the magnetic poles are situated is $z_0 = 2.4$. We also notice that with increasing accuracy in the $\theta$ grid from $N = 15$ to $N = 25$, the two MAP solutions ($m = 1$, $s = 1$ and $m = 2$, $s = 0$) appear to approach each other from different direction, Fig. (2). Hence it is interesting to know whether the two will coincide when the accuracy in $r$ and $\theta$ grid is high enough. This will be done with more powerful computer in a later work.

## VI. CONCLUSION

Following results in Ref. [7] where it is possible to alternatively describe MAP and MAC solutions in terms of integer $s$, and winding number $m = 1$, we study such axially symmetric monopole-antimonopole pair solutions, parameterized by integer $m = 1$, $s = 1$ and $n = 1$. This solution resides in the topological vacuum sector and it represents a monopole on the positive $z$-axis and antimonopole on the negative $z$-axis. The configuration hence possesses zero net magnetic charge.

This MAP solution is similar to the MAP solutions of Kleihaus and Kunz [5]. Our MAP solution has $m = 1$, $s = 1$ and $z_0 = 2.4$ for $M = 40$, $N = 25$. When the accuracy in the $\theta$ grid is increased from $N = 15$ to $N = 25$, the two MAP solutions ($m = 1$, $s = 1$ and $m = 2$, $s = 0$) appear to approach each other from different direction. Questions of whether these two numerical solutions are the same configuration will be answered with more powerful computers and will be addressed elsewhere. It is also interesting to compare the energies of the two systems of MAPs and this will be calculated in another paper.

The authors would like to thank the Academy of Sciences, Malaysia, for the Scientific Advancement Grant Allocation, SAGA (304/PFIZIK/653004/A118).

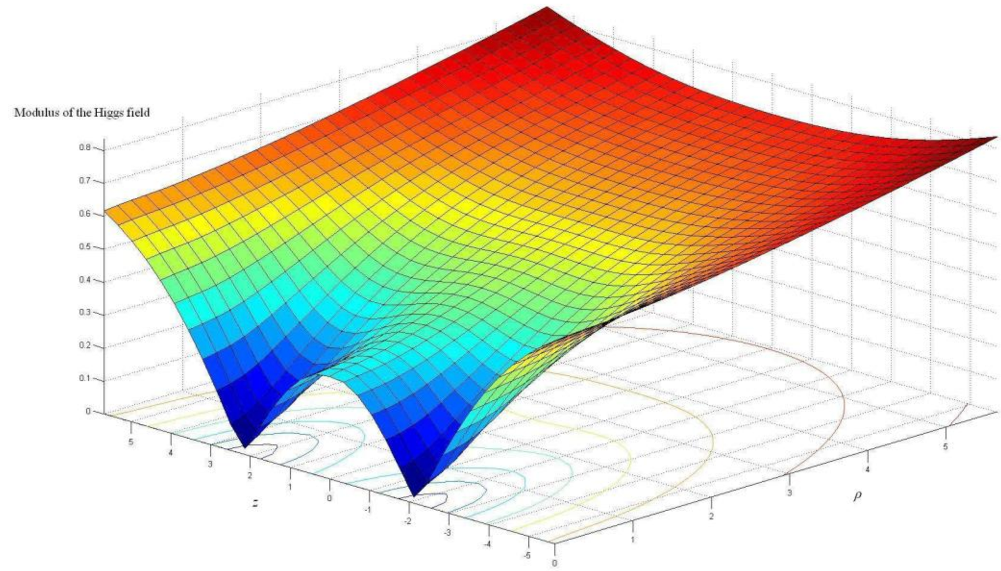


**FIG. 1.** The modulus of the Higgs field as a function of $\rho$ and $z$.

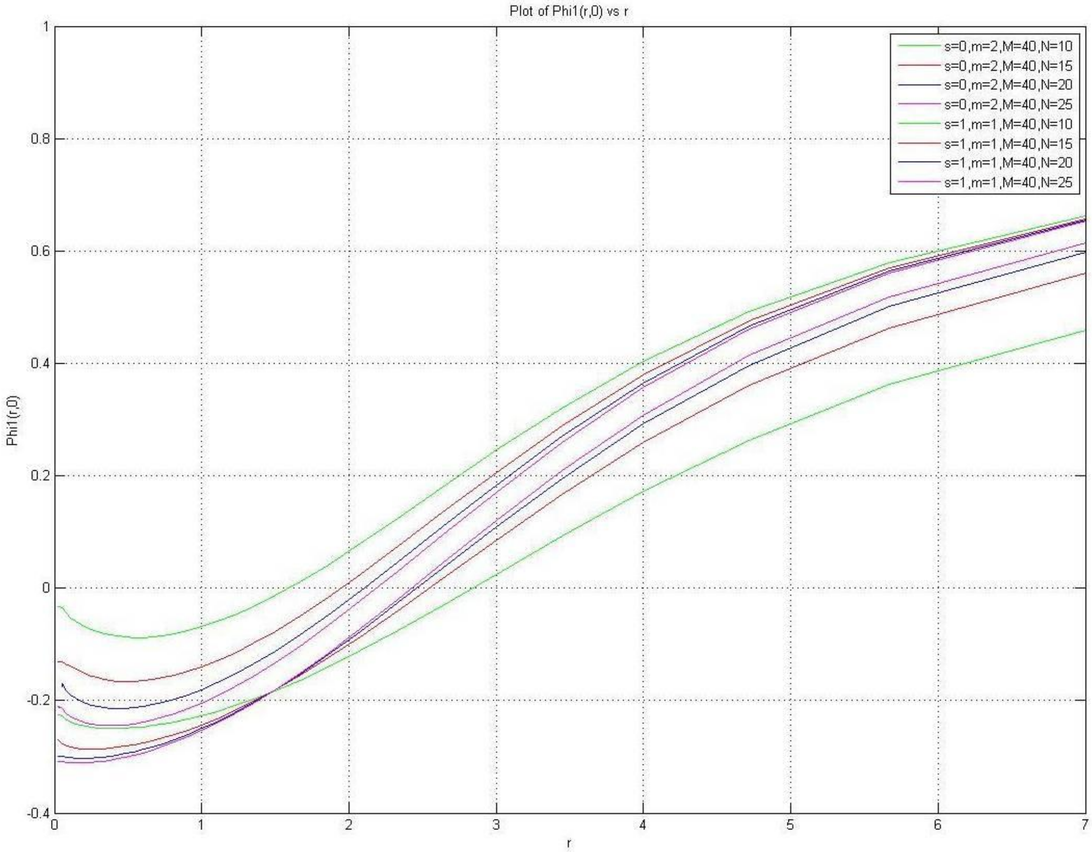


**FIG. 2.** A plot of $\Phi = (r,0)$ versus $r$ for $M$ = 40, $N$ = 10, 15, 20, 25 for the MAP solutions parameterized by $m$ = 2, $s$ = 0, $n$ = 1 (the first four graph on the left) and $m$ = 1, $s$ = 1, $n$ = 1 (the next four graphs on the right).

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