Generalized chaos synchronization of discrete maps via linear transformations

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A generalized approach for constructing chaotically synchronized discrete dynamical systems via linear transformation is proposed in this paper. The generalized Henon map is treated as an example. The proposed method is simpler than the conventional methods because here we need not required to calculate the Lyapunov exponents. In our method the functional relationship between the states of the driving system and driven system can be determined. Therefore knowing the driven system the behavior of the drive system can be predicted in advance here. This synchronization scheme may be used for sending secret message for the purpose of communications.

I. INTRODUCTION

Experimental observations have pointed out that chaotic systems are common in nature. It is found that in Chemistry (Belouov-Zhabotinski reaction), in Nonlinear Optics (lasers), in Electronics (Chua-Matsumoto circuits), in Fluid Dynamics (Rayleigh-Benard convection), etc. chaotic systems exist. Chaos is found in meteorology, solar system, heart and brain of living organisms and so on. Synchronization and control of interacting chaotic oscillators is one of the fundamental phenomena of nonlinear dynamics and chaos. Experimental realization of chaos Synchronization and control have been achieved with a magnetoelastic ribbon, a heart, a thermal convection loop, a yttrium iron garnet oscillator, a diode oscillator, an optical multimode chaotic solid-state laser, a Belousov-Zhabotinski reaction diffusion chemical system, and many other experiments.

Since Pecora and Carroll's [1] work many effective methods namely, OGY method [2], adaptive control [3], differential geometric method [4], inverse optimal control [5], lag synchronization [6], projective synchronization [7], etc. for chaos control and synchronization have been proposed. Usually two dynamical systems are called synchronized if the distance between their corresponding states converges to zero as time goes to infinity. This type of synchronization is known as identical synchronization [1]. A generalization of the concept for unidirectionally coupled dynamical systems was proposed by Rulkov *et al*. [8], where two systems are called synchronized if a static functional relationship exists between the states of the systems. They called this kind of synchronization a generalized synchronization (GS). Kocarev and Parlitz [9] formulated a condition for the occurrence of GS between two coupled continuous dynamical systems. Yang and Chua [10] proposed GS of continuous dynamical systems via linear transformations. In this letter we proposed a generalized approach for constructing chaotically synchronized discrete dynamical systems via linear transformations. The method is based on the stability criterion of linear systems. Without calculating the Lyapunov exponents of the constructed system the linear GS can be robustly achieved. The main advantage of the proposed method is that it can provide satisfactory GS for autonomous and nonautonomous systems. Here the autonomous generalized Henon map is treated as example. In projective synchronization $\lim_{n\to\infty} \frac{y_i(n)}{x_i(n)} = \alpha$ where α is a real constant. Clearly, the projective synchronization can be obtained as a special case of the GS.

II. GENERALIZED SYNCHRONIZATION VIA LINEAR TRANSFORMATION

Any discrete dynamical system can be decomposed into two parts

$$X(k+1) = AX(k) + \Psi X(k) \tag{1}$$

where $X = (x_1, x_2, x_3, \ldots, x_n)^t$ and A is an $n\times n$ constant matrix and $\Psi : R^n \to R^n$. We assume that the drive system transmit the signal $\Psi(X)$ to the response system and consider the following unidirectional coupling scheme:

$$\begin{aligned} X(k+1) &= AX(k) + \Psi X(k) \\ Y(k+1) &= AY(k) + \Psi Y(k) \end{aligned} \tag{2}$$

where Λ is a $n\times n$ matrix. Notice that the marix Λ may be a time dependent matrix, not necessarily a constant matrix.

Theorem: If the matrix Λ commutes with *A***,** the two dynamical systems are in a state of generalized synchronization via the following generalized synchronization transformation:

$$Y(\infty) = H(X) + X$$

if and only if *A* has spectral radius less than 1. i.e, if all eigenvalues of the matrix *A* has modulus less than 1.

Proof. Let $Z(k) = Y(k) - \Lambda(k) - \Lambda X(k)$**.** Then the stability of the generalized synchronization between the two dynamical systems via the generalized transformation $Y(k) = H(k) = X(k)$ is equivalent to that of the origin of the following system:

$$\begin{aligned} Z(k+1) &= Y(k+1) - X(k+1) \\ &= AY(k) + \Lambda\Psi(k) - \Lambda[AX(k) + \Psi(X(k))] \\ &= A[Y(k) - \Lambda X(k)] \text{ since } \Lambda \text{ commutes with } A \\ &= AZ(k). \end{aligned} \tag{3}$$

Therefore $\lim_{k\to\infty} Z(k) = 0$ if and only if all eigenvalues (real or complex) of the matrix *A* having modulus less than 1. Therefore the matrix *A* can be taken as any real matrix with all eigen values having modulus less than 1. So there are infinite ways to choose the matrix *A*.

The matrices *P* which commute with a $n \times n$ matrix which satisfies the following equation

$$AP = PA. \tag{4}$$

Clearly the above equation has infinite number of solutions, therefore we can construct several methods of linear generalized synchronization between two chaotic systems.

III. GENERALIZED SYNCHRONIZATION OF GENERALIZED HENON MAPS

In this section we discuss the GS of two discrete time chaotic generalized Henon map via linear transformation. We consider the Henon map as

$$\begin{aligned} x_1(k+1) &= -bx_3(k) \\ x_2(k+1) &= bx_3(k) + x_1(k) \\ x_3(k+1) &= 1 + x_2(k) - ax_3^2(k) \end{aligned} \tag{5}$$

where *a* and *b* are parameters. For $a = 1.07$ and $b = 0.3$ the Henon map displays a chaotic attractor. The Henon map can be decomposed into two parts as

$$X(k+1) = AX(k) + \Psi(X(k)) \tag{6}$$

in many ways. Here we consider the two types of decompositions. Firstly we consider

$$A = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \tag{7}$$

$X = (x_1, x_2, x_3)^t$ and $\Psi(X) = [-a_1x_1(k) - bx_3(k), x_1 - a_2x_2(k) + bx_3(k), 1 + x_2(k) - a_3x_3(k) - a_3x_3^2(k)]^t$. Clearly the matrix *A* has eigenvalues which are all less than 1 in modulus if and only if $\alpha_1 < 1$, $\alpha_2 < 1$ and $\alpha_3 < 1$.

Lastly we take

$$A = \begin{pmatrix} \alpha_1 & 0 & -b \\ 0 & \alpha_2 & b \\ 0 & 0 & \alpha_3 \end{pmatrix} \tag{8}$$

$X = (x_1, x_2, x_3)^t$ and $\Psi(X) = [-a_1x_1(k), x_1(k) - a_2x_2(k), 1 + x_2(k) - a_3x_3(k) - ax_3^2(k)]^t$. Clearly the matrix *A* has eigenvalues which are all less than 1 in modulus if and only if $\alpha_1 < 1$, $\alpha_2 < 1$ and $\alpha_3 < 1$.

Now the driven Henon map can be taken as

$$Y(k+1) = AY(k) + \Lambda\Psi(X(k)) \tag{9}$$

where the matrix Λ commutes with *A*. Therefore the driving Henon map and driven Henon map will synchronize in the generalized sense.

IV. RESULTS AND DISCUSSIONS .

We did the numerical simulation of the generalized Henon map taking random initial conditions for both the driving and driven system. First two simulations are done for decomposition of the Henon map as in Eq. (7) and rest three simulations are done using decomposition of the Henon map as in Eq. (8).

Simulation 1.

In this simulation, we take

$$\Lambda = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \tag{10}$$

where λ is real. Obviously the sufficient condition for GS $\Lambda A = A\Lambda$ is satisfied. Here the driving Henon map is given by Eq. (5) and the driven Henon map is given by

$$y_1(k+1) = -\alpha_1 y_1(k) - \lambda\alpha_1 x_1(k) - \lambda b x_3(k)$$
$$y_2(k+1) = \alpha_2 y_2(k) - \lambda(x_1(k) - \alpha_2 x_2(k) + b x_3(k))$$
$$y_3(k+1) = \alpha_3 y_3(k) - \lambda(1 + x_2(k) - \alpha_3 x_3(k) + b x_3^2(k)). \quad (11)$$

In this case we define the generalized synchronization errors as $e_1(k) = y_1(k) - \frac{x_1(k)}{2}$, $e_2(k) = y_2(k) - \frac{x_2(k)}{2}$ and $e_3(k) = y_3(k) - \frac{x_3(k)}{2}$. Taking $\alpha_1 = 0.9$, $\alpha_2 = 0.8$ and $\alpha_3 = 0.75$ the time evaluation of the generalized synchronization errors $e_1(t)$, $e_2(t)$ and $e_3(t)$ are shown in Fig. 1. In this case the state variables of the driving system and the driven system are connected by the linear transformations

$$y_1(k) - \frac{x_1(k)}{2}$$
$$y_2(k) - \frac{x_2(k)}{2} \quad (12)$$
$$y_3(k) - \frac{x_3(k)}{2}.$$

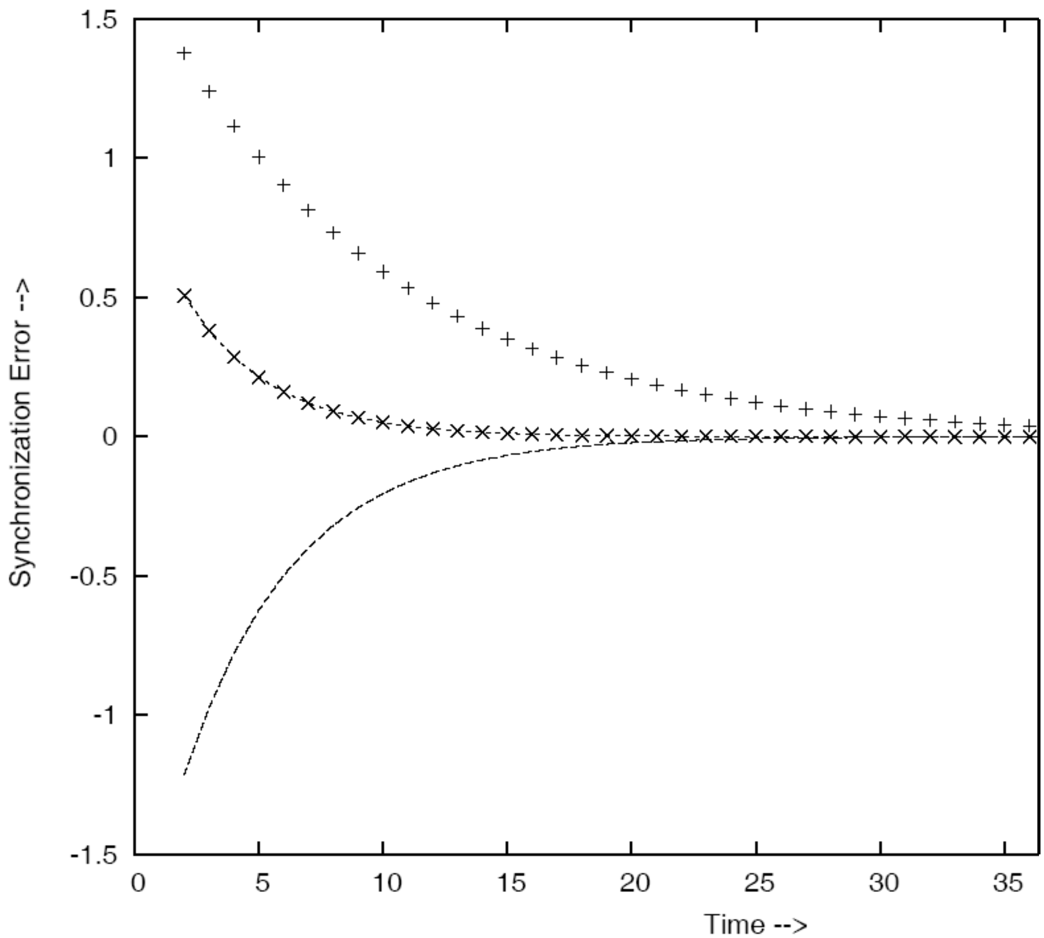

FIG. 1. Time evolution of the generalized synchronization errors $e_1(t)$, $e_2(t)$ and $e_3(t)$ are shown for $\alpha_1 = 0.9$, $\alpha_2 = 0.8$ and $\alpha_3 = 0.75$.

Simulation 2.

In this simulation, we take

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad (13)$$

where λ_1, λ_2, λ_3 are all real. In this case the driven system is given by,

$$\begin{aligned} y_1(k+1) &= \alpha_1 y_1(k) - \lambda_1(bx_3(k) - \alpha_1 x_1(k)) \\ y_2(k+1) &= \alpha_2 y_2(k) - \lambda_2(x_1(k) - \alpha_2 x_2(k) + bx_3(k)) \\ y_3(k+1) &= \alpha_3 y_3(k) - \lambda_3(1 + x_2(k) - \alpha_3 x_3(k) + \alpha x_3^2(k)). \end{aligned} \quad (14)$$

In this case we define the generalized synchronization errors as $e_1(k) = y_1(k) - x_1(k)$, $e_2(k) = y_2(k) - 2x_2(k)$ and $e_3(k) = y_3(k) - 3x_3(k)$. Taking $\alpha_1 = 0.85$, $\alpha_2 = 0.75$ and $\alpha_3 = 0.95$ the time evaluation of the generalized synchronization errors $e_1(k)$, $e_2(k)$ and $e_3(k)$ are shown in Fig. 2.

For $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ the state variables of the driving system and the driven system are connected by the linear transformations

$$\begin{aligned} y_1(k) &= x_1(k) \\ y_2(k) &= 2x_2(k) \\ y_3(k) &= 3x_3(k). \end{aligned} \quad (15)$$

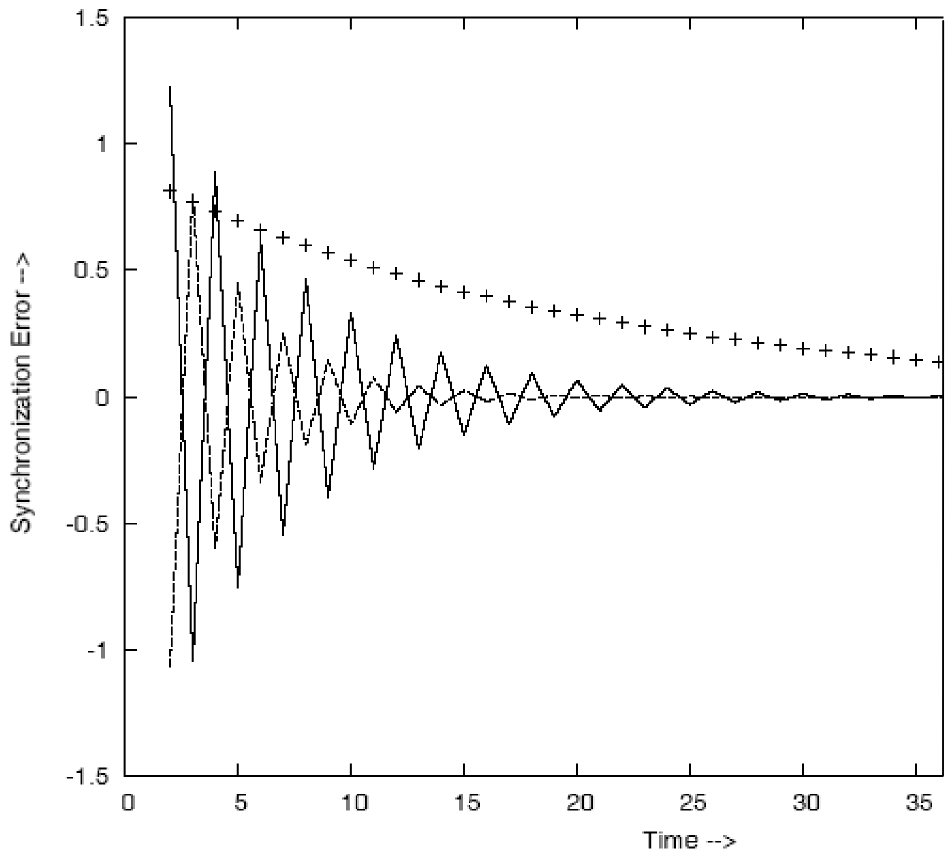

FIG. 2. Time evolution of the generalized synchronization errors $e_1(t)$, $e_2(t)$ and $e_3(t)$ are shown for $\alpha_1 = 0.85$, $\alpha_2 = 0.75$ and $\alpha_3 = 0.95$.

Simulation 3.

In this simulation, we take

$$\Lambda = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad (16)$$

where λ is real. In this case the driven system is given by

$$\begin{aligned} y_1(k+1) &= \alpha_1 y_1(k) - by_3(k) - \lambda\alpha_1 x_1(k) \\ y_2(k+1) &= \alpha_2 y_2(k) - \lambda(x_1(k) - \alpha_2 x_2(k) + by_3(k)) \\ y_3(k+1) &= \alpha_3 y_3(k) + \lambda(1 + x_2(k) - \alpha_3 x_3(k) + \alpha x_3^2(k)). \end{aligned} \quad (17)$$

In this case we define the generalized synchronization errors as $e_1(k) = y_1(k) - 2x_1(k)$, $e_2(k) = y_2(k) - 2x_2(k)$ and $e_3(k) = y_3(k) - 2x_3(k)$ .Taking $\alpha_1 = 0.5$, $\alpha_2 = 0.95$ and $\alpha_3 = 0.70$ the time evaluation of the generalized synchronization errors $e_1(k)$, $e_2(k)$ and $e_3(k)$ are shown in Fig. 3.

For $\lambda_2 = 2$ the state variables of the driving system and the driven system are connected by the linear transformations

$$\begin{aligned} y_1(k) &= 2x_1(k) \\ y_2(k) &= 2x_2(k) \\ y_3(k) &= 2x_3(k). \end{aligned} \quad (18)$$

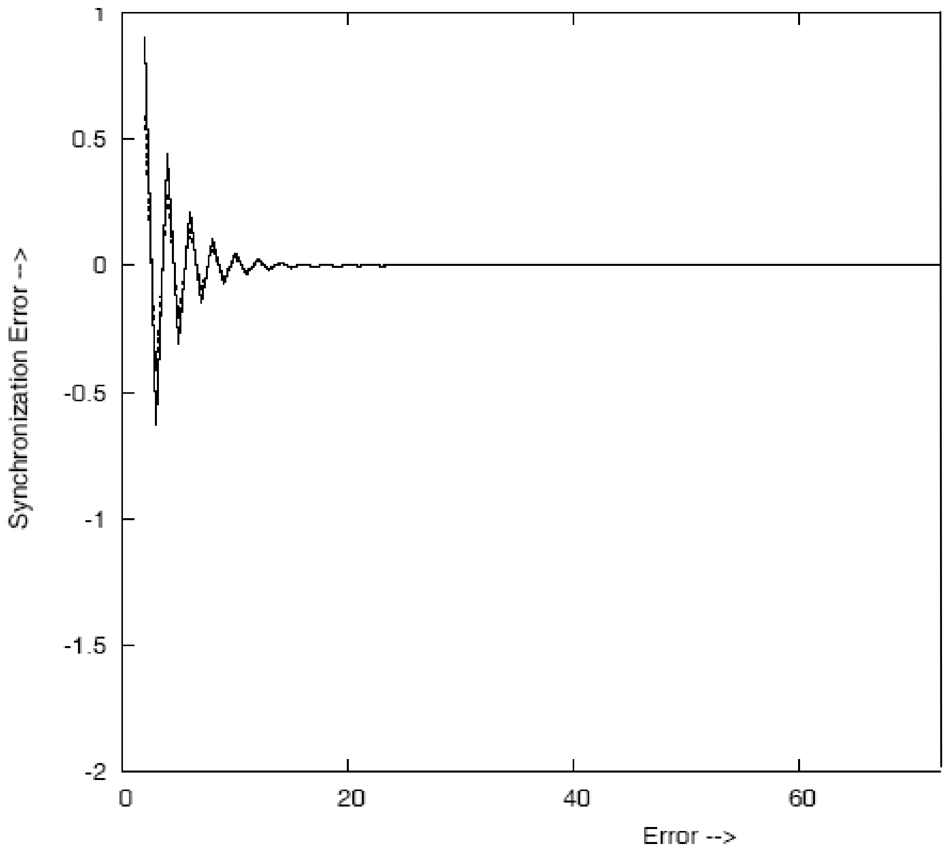

FIG. 3. Time evolution of the generalized synchronization errors $e_1(t)$, $e_2(t)$ and $e_3(t)$ are shown for $\alpha_1 = 0.5$, $\alpha_2 = 0.95$ and $\alpha_3 = 0.70$.

Simulation 4.

In this simulation, we take

$$\Lambda = \begin{pmatrix} \lambda & 0 & -b \\ 0 & \lambda & b \\ 0 & 0 & \lambda \end{pmatrix}. \quad (19)$$

In this case the driven Henon system is given by

$$\begin{aligned} y_1(k+1) &= \alpha_1 y_1(k) - b y_3(k) - \alpha_1^2 x_1(k) \\ &\quad - b(1 + x_2(k) - \alpha_3 x_1(k) + \alpha x_3^2(k)) \\ y_2(k+1) &= \alpha_2 y_2(k) + b y_3(k) + \alpha_2 (x_1(k) - \alpha_2 x_2(k)) \\ &\quad + b(1 + x_2(k) - \alpha_3 x_3(k) + \alpha x_3^2(k)) \\ y_3(k+1) &= \alpha_3 y_3(k) + \alpha_3 (1 + x_2(k) - \alpha_3 x_3(k) + \alpha x_3^2(k)). \end{aligned} \quad (20)$$

In this case we define the generalized synchronization errors as $e_1(k) = y_1(k) - \alpha_1 x_1(k) + b x_3(k)$, $e_2(k) = y_2(k) - \alpha_2 x_2(k) - b x_3(k)$ and $e_3(k) = y_3(k) - \alpha_3 x_3(k)$. Taking $\alpha_1 = 0.85$, $\alpha_2 = 0.75$ and $\alpha_3 = 0.95$ the time evaluation of the generalized synchronization errors $e_1(t)$, $e_2(t)$ and $e_3(t)$ are shown in Fig. 4.

In this case the state variables of the driving system and the driven system are connected by the linear transformations

$$\begin{aligned} y_1(k) &= \alpha_1 x_1(k) - b x_3(k) \\ y_2(k) &= \alpha_2 x_2(k) + b x_3(k) \\ y_3(k) &= \alpha_3 x_3(k). \end{aligned} \quad (21)$$

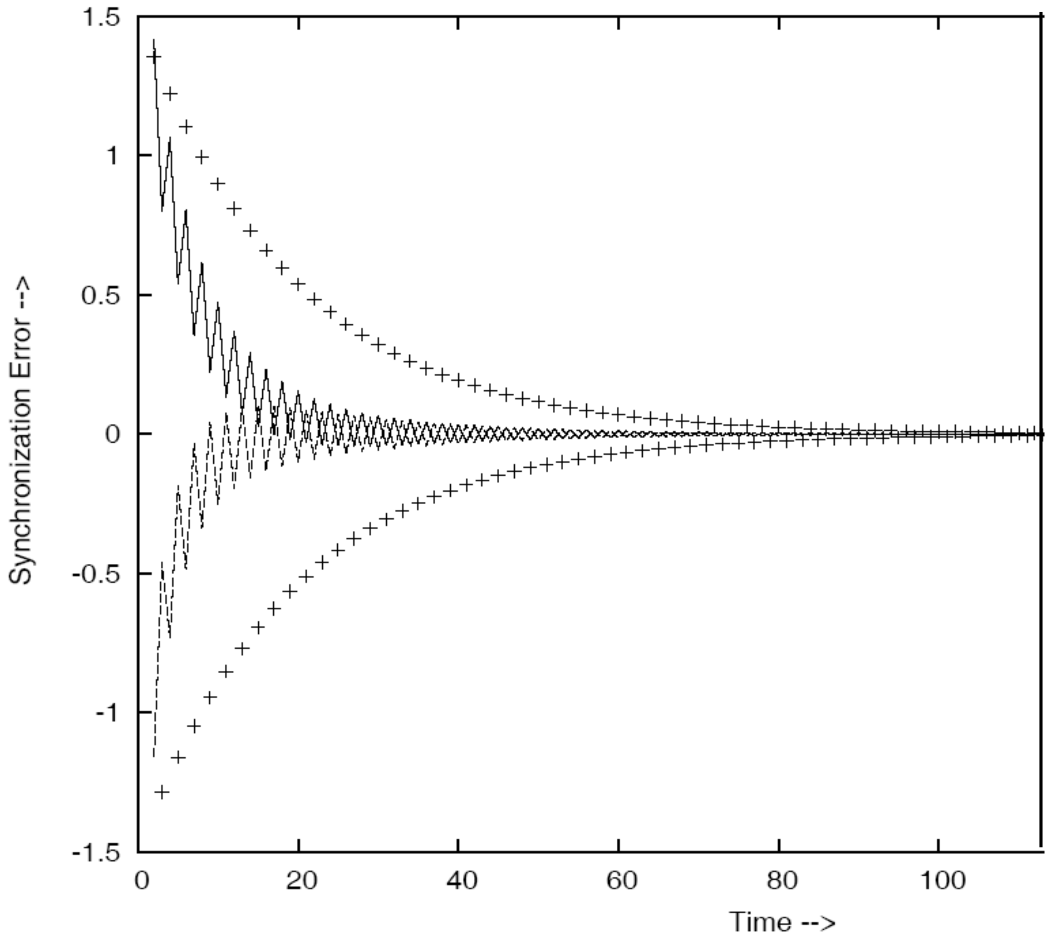

FIG. 4. Time evolution of the generalized synchronization errors $e_1(t)$, $e_2(t)$ and $e_3(t)$ are shown for $\alpha_1 = 0.85$, $\alpha_2 = 0.75$ and $\alpha_3 = 0.95$.

Simulation 5.

In this simulation, we take

$$\Lambda = A^{-1} = \begin{pmatrix} \alpha_1 & 0 & \dfrac{b}{\alpha_1\alpha_3} \\ 0 & \alpha_2 & \dfrac{-b}{\alpha_1\alpha_3} \\ 0 & 0 & \alpha_3 \end{pmatrix}. \tag{22}$$

In this case the driven Henon system is given by

$$\begin{aligned} y_1(k+1) &= \alpha_1 y_1(k) - by_3(k) - x_1(k) \\ &\quad - \frac{b}{\alpha_1\alpha_3}(1 + x_2(k) - \alpha_3 x_3(k) + \alpha x_3^2(k)) \\ y_2(k+1) &= \alpha_2 y_2(k) + by_3(k) + \frac{1}{\alpha_2}(x_1(k) - \alpha_2 x_2(k)) \\ &\quad + \frac{b}{\alpha_2\alpha_3}(1 + x_2(k) - \alpha_3 x_3(k) + \alpha x_3^2(k)) \\ y_3(k+1) &= \alpha_3 y_3(k) + \frac{1}{\alpha_3}(1 + x_2(k) - \alpha_3 x_3(k) + \alpha x_3^2(k)). \end{aligned} \tag{23}$$

In this case we define the generalized synchronization errors as $e_1(k) = y_1(k) - \left[\frac{x_1(k)}{\alpha_1} - \frac{bx_3(k)}{\alpha_2\alpha_3}\right]$, $e_2(k) = y_2(k) - \left[\frac{x_2(k)}{\alpha_2} - \frac{bx_3(k)}{\alpha_2\alpha_3}\right]$ and $e_3(k) = y_3(k) - \frac{x_3(k)}{\alpha_3}$. Taking $\alpha_1 = 0.9$, $\alpha_2 = 0.9$ and $\alpha_3 = 0.95$ the time evaluation of the generalized synchronization errors $e_1(t)$, $e_2(t)$ and $e_3(t)$ are shown in Fig. 5.

In this case the state variables of the driving system and the driven system are connected by the linear transformations

$$\begin{aligned} y_1(k) &= \frac{1}{\alpha_1}x_1(k) - \frac{b}{\alpha_1\alpha_3}x_3(k) \\ y_2(k) &= \frac{1}{\alpha_2}x_2(k) - \frac{b}{\alpha_2\alpha_3}x_3(k) \\ y_3(k) &= \frac{1}{\alpha_3}x_3(k). \end{aligned} \tag{24}$$

V. CONCLUSIONS

A generalized approach for constructing chaotically synchronized discrete dynamical systems via linear transformation is proposed in this paper. The proposed method is simpler than the conventional methods because here we need not required to calculate the Lyapunov exponents. In our method the functional relationship between the states of the driving system and driven system can be determined. Therefore knowing the driving system the behavior of the driven system can be predicted in advance here. If the matrix associated with the linear transformation is invertible then from the behavior of the driven system it is possible to predict the behavior of the driving system.

This synchronization scheme may be used for sending secret message for the purpose of communications. For the communication purpose our goal is to get information of the drive system through a response system, then we can construct a response system which will synchronize with the driving system. For communication purpose GS is more effectively secret because for the outsiders who want to extract information transmitted by the driving system, apart from knowing the mechanism of the drive system and the way of driving signal operators, they have to know another code "the functional relationship between the variables of the drive and response system". In GS there exist infinite ways to choose the secrete key. Therefore, the techniques based on GS is seemed to be more practical in secrete communication applications.

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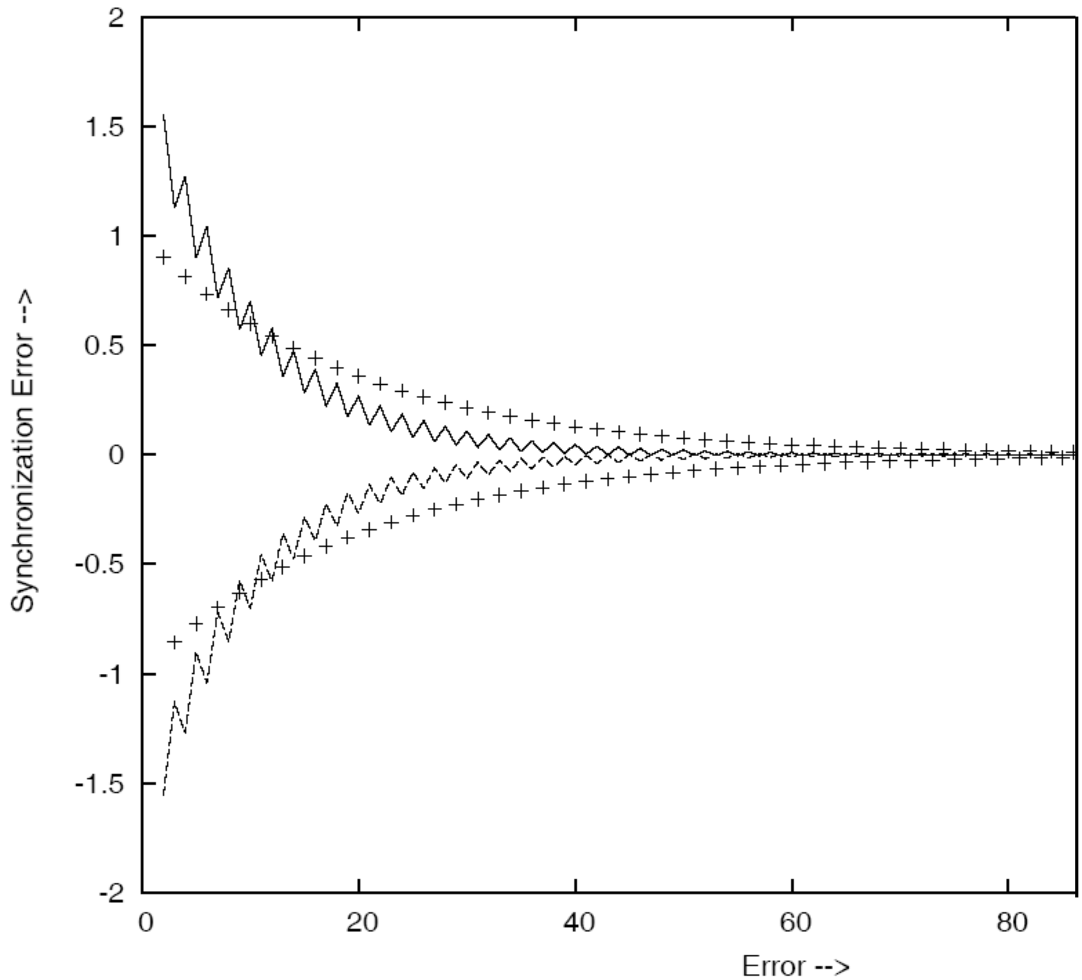

FIG. 5. Time evolution of the generalized synchronization errors $e_1(t)$, $e_2(t)$ and $e_3(t)$ are shown for $\alpha_1 = 0.9$, $\alpha_2 = 0.9$ and $\alpha_3 = 0.95$.