Numerical relationship of Metallic Photonic Crystals' Effective Plasma Frequency with Relative Permittivity

Low Khee Lam[1,*] and Mohd Zubir Mat Jafri[2,*]

[1]*KDU Penang University College, Penang, 10400 Malaysia*
[2]*Universiti Sains Malaysia, Penang, 11700 Malaysia*

* *kheelam.low@kdupg.edu.my and mjafri@usm.my*

(Received: 24 August 2016; published XX September 2016)

Abstract. Several analytical models have been used to investigate the effective plasma frequency of photonic crystals involving metallic components. In recent studies, the analysis models were not suitable for revealing detailed band structure effects. Furthermore, the effect of the relative permittivity used in photonic crystals has not been discussed and related to the analytical model. By using the modified plane wave expansion method in the literature, we have plotted band structure graphs for photonic crystals containing metallic components and photonic crystals in metallic media. We investigated two metals: copper ($\omega_p = 1914\ \text{THz}$) and aluminum ($\omega_p = 3570\ \text{THz}$). We obtained theoretical values of the effective plasma frequency by varying the relative permittivity and filling fraction using band structure graphs. Then we obtained a numerical model of effective plasma frequency and relative permittivity using regression method for both copper and aluminum. Both shared the same equation. Next the derivation was expanded to photonic crystals in metallic media, going through the same process. A numerical model of effective plasma frequency and relative permittivity was obtained for photonic crystals in metallic media.

Keywords: Photonic crystals, frequency dependent material, effective plasma frequency, and regression method.

I. INTRODUCTION

Photonic crystals were first introduced by Yablonivitch [1] and Sajeev [2] in 1987. Then, the theory of photonic crystals was expanded from frequency-independent materials to frequency-dependent materials. The energy band structure is very important for photonic crystals. It can be used for investigation of fundamental properties. Just as the periodicity of solid state crystals determines the energy band structure, the structuring of photonic crystals at wavelength scales has turned out to be a viable approach for control of the photons.

Using frequency-dependent materials in photonic crystals will show different characteristics of photonic crystals. Negative refraction index [3], effective plasma frequency [4–6], and surface plasmon effect [7, 8] can be observed using the band structure graphs of photonic crystals. Plasma frequency is a basic property of frequency-dependent materials and it cannot be changed. We have used the Drude model to describe the dielectric constant of metal as follows:

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega - i\gamma)} \tag{1}$$

where ω_p is the plasma frequency and γ is damping constant. Damping constant is neglected because it is small compared to plasma frequency.[7] So, we obtained a simple Drude model which the imaginary part of equation 1 is neglected.

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \tag{2}$$

It limits wave propagation in metals. From the ordinary metallic band graph, effective plasma frequency is the frequency at the minimum of the lowest band in a band energy graph [14]. Pendry et al. [6, 9] considered the problem of the plasma frequency of an array of thin metallic wires and showed that the effective plasma frequency can be tailored at a lower frequency, at which a propagating electromagnetic mode does not exist. So, several researchers [6, 9–12] have proposed an analytical model of an effective plasma frequency for photonic crystals containing a metallic component. Unfortunately, the analytical models are too simple to explain the effective plasma frequency [4]. The analytical models of effective plasma frequency included only the physical dimensions of photonic crystals such as radius and lattice constant. Moreover, the analytical models are valid only for metallic components in vacuum.

The effective plasma frequency can be derived from the group velocity, or by differentiation of the energy band equation of photonic crystals; it is the zero value of the group velocity. Kuzmiak et al. [13] derived an analytical equation for group velocity, in which photonic crystals contain a frequency-dependent component. But the derivation was limited to frequency-dependent properties and vacuum properties. There was no relative permittivity included in the structure. Moreover, the effect of photonic crystals using metallic media has not been discussed in any prior publications.

In this article, we used the plane wave expansion method, modified for application to metallic media [14, 15], to investigate metallic photonic crystals including dielectric components for the E polarization mode for square lattices. Realistic cases were investigated. We used copper and aluminum as our frequency-dependent materials and varied the relative permittivity from 1 to 12.96. Then, the effective plasma frequencies were obtained from the band structure graphs.Determining the relationship of relative permittivity of rods or background to the effective plasma frequency was a goal of this study. So, a numerical model for both photonic crystals containing metallic components and photonic crystals in metallic media using statistical analysis was derived. The investigation was then expanded to triangular lattices to study the effect of lattice arrangement on the effective plasma frequency.

II. METHOD

Two cases were considered: Case I, photonic crystals containing metallic components; Case II, photonic crystals in metallic media. The two equations for these cases include the relative permittivity of frequency-independent materials [14, 15]. In the literature, calculations have involved the E and H polarizations. But for photonic crystals in H polarization, the effective plasma frequency starts from zero, whereas in E polarization it starts from a minimum value. So, values in E polarization draw our interest. By using the modified plane wave expansion method, the band structure graph equation for the E polarization mode can be simplified as below.

Case I:

$$\mu^2 \varepsilon B(k|G) + \mu^2 f\left(1-\varepsilon\right)\sum_{G'}\frac{2J_1|(G-G')|R}{|(G-G')|R}B(k|G')$$
$$-\left[f\sum_{G'}\left[\frac{\omega_p^2}{c^2}\right]\frac{2J_1|(G-G')|R}{|(G-G')|R}B(k|G') + (k+G)^2 B(k|G)\right]=0 \tag{3}$$

Case II:

$$\mu^2 B(k|G) + \mu^2\sum_{G'}\left(\varepsilon-1\right)\frac{2fJ_1(|G-G'|R)}{|G-G'|R}B(k|G') - \frac{\omega_p^2}{c^2}B(k|G)$$
$$-\sum_{G'}\frac{\omega_p^2}{c^2}\frac{2fJ_1(|G-G'|R)}{|G-G'|R}B(k|G') - \left(k+G\right)^2 B\left(k|G\right)=0 \tag{4}$$

where $\mu=\omega/c$, ε is the relative permittivity, k is the wave vector, G is the reciprocal lattice, J_1 is the first Bessel function, R is the radius of rods, ω_p is the plasma frequency of the metals, c is the speed of light, and ω is the eigenfrequency. Equation 3 and 4 can be solved by using a linearization technique. Figure 1 is an example of a band structure graph for Case I in which the metals rods are copper ($\omega_p = 1914$ THz) embedded in a gallium arsenide medium ($\varepsilon = 12.96$) for a square lattice. The effective plasma frequency is the lowest value of the green line at position Γ. The normalized frequency is 0.0876. Figure 2 is an example of a band structure graph for Case II which the gallium arsenide rods ($\varepsilon = 12.96$) are embedded in a copper

($\omega_p = 1914$ THz) medium for a square lattice. The lowest value of the green line at Γ is 1.1229, which is the effective plasma frequency. So, the same method is used for Case I and Case II to collect all the effective plasma frequencies for copper but with variation of the relative permittivity from 1 to 12.96.

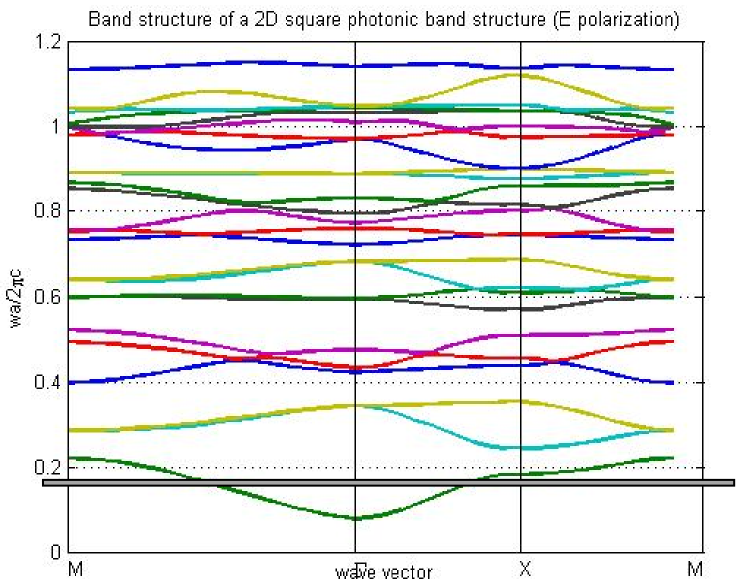

FIGURE 1. Photonic band structure of photonic crystals with copper rods in GaAs ($\varepsilon = 12.96$) at $f = 0.5$ for the E polarization mode. Shaded area is the band gap.

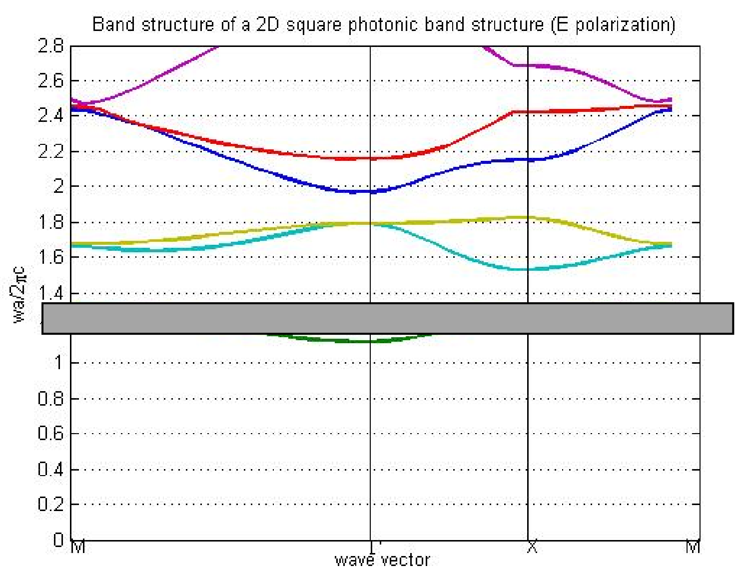

FIGURE 2. Photonic band structure for photonic crystals of GaAs rods ($\varepsilon = 12.96$) in copper slab at $f = 0.5$ for E polarization. Shaded area is the band gap.

III. RESULTS AND DISCUSSIONS

Case I

Using the method suggested in the previous section, all the values of effective plasma frequency of photonic crystals with lattice constant 1 μm were collected for square lattices and triangular lattices. The effective plasma frequencies for copper and aluminum are shown in Figures 3 and 4 for square lattices. Both graphs show similar patterns but with different values. This shows that the effective plasma frequencies have similar values although different metals are used. Figures 5 and 6 are for the same materials but in a different arrangement, namely a triangular lattice. Both graphs show the same pattern.

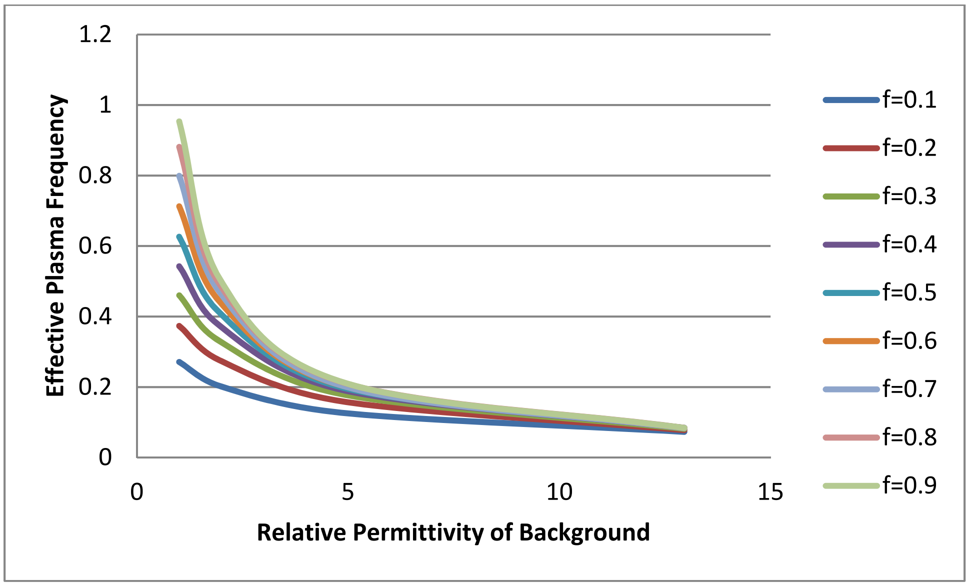

FIGURE 3. Effective plasma frequency versus relative permittivity for photonic crystals containing copper rods for square lattice arrangement with different filling fractions.

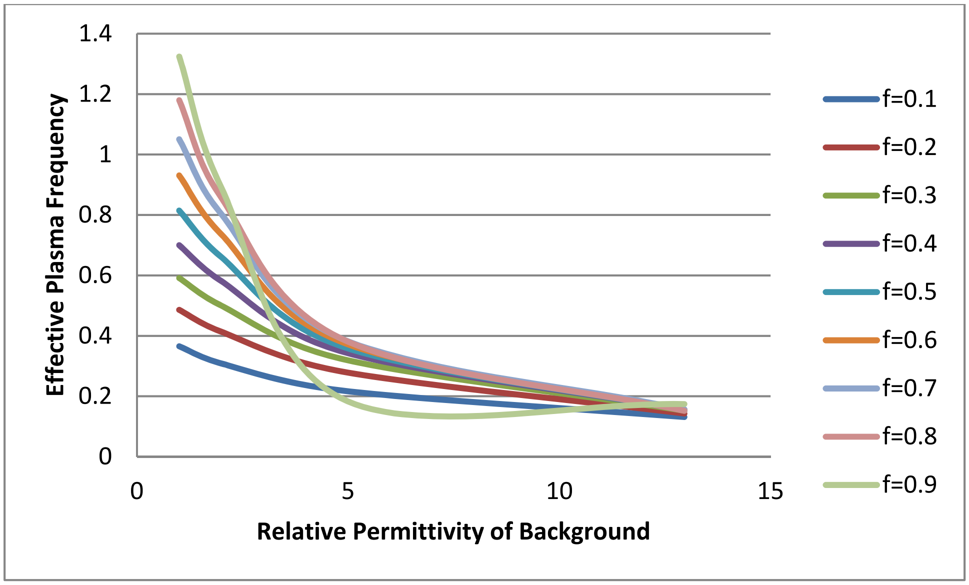

FIGURE 4. Effective plasma frequency versus relative permittivity for photonic crystals containing aluminum rods for square lattice arrangement with different filling fractions.

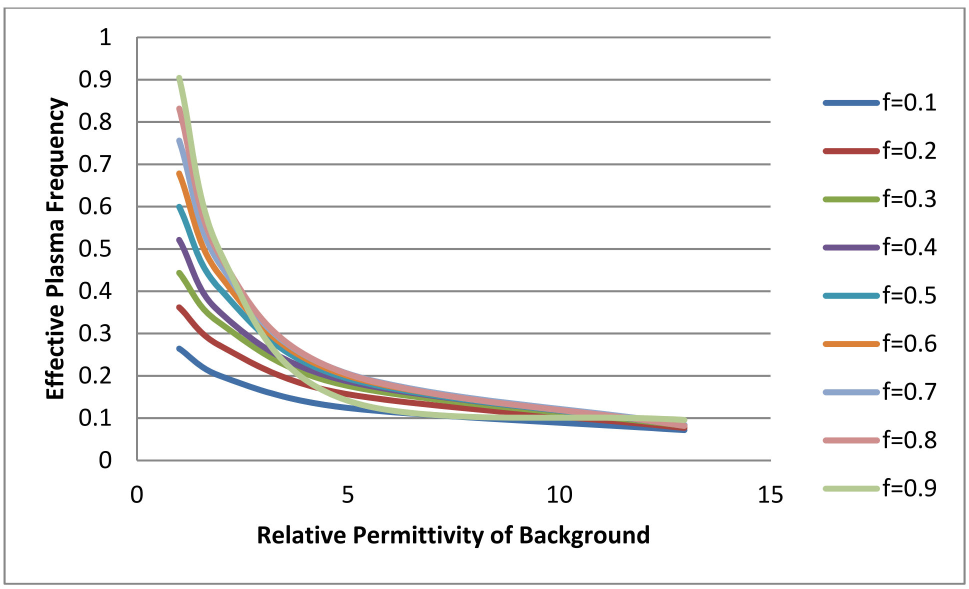

FIGURE 5. Effective plasma frequency versus relative permittivity for photonic crystals containing copper rods for triangular lattice arrangement with different filling fractions.

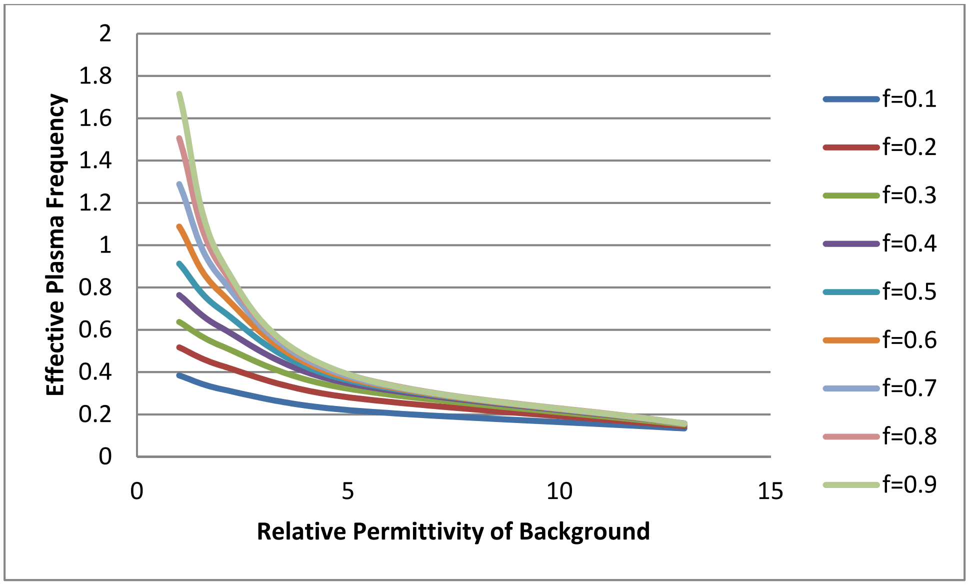

FIGURE 6. Effective plasma frequency versus relative permittivity for photonic crystals containing aluminum rods for triangular lattice arrangement with different filling fractions.

Figures 3 and 5 show similar values for both square lattice and triangular lattices for each metal as well as Figures 4 and 6. This means that the effective plasma frequency is not affected by the lattice arrangement or wave vector. But the size of the rods does affect the effective plasma frequency. This is because when different filling fractions are used, different graph patterns are produced. Figures 3–6 show exponential decay graph patterns. The effective plasma frequency is high for a low relative permittivity background but it will decrease to 90% of the original plasma frequency for a high relative permittivity. All the values of effective plasma frequency for all the filling fractions will meet at that point for high relative permittivity. This indicates that for high relative permittivity the effective plasma frequency is not affected by the filling fraction. So, by using the polynomial regression method in SigmaPlot®, a the relationship from the Figure 6 is shown in equation 5.

$$\upsilon_{\text{eff}} = ae^{-b\varepsilon_0} \tag{5}$$

where *a* and *b* are constant parameters. This model produces a high correlation between the theoretical data and statistical data. Table 1 shows the change of parameters when filling fraction is changed and also the root mean square of the correlation between equation 5 and the data from SigmaPlot®. The parameters *a* and *b* change according to the filling fraction. So, the parameters *a* and *b* depend on the physical parameters of the photonic crystals such as radius of rods and lattice constant. Copper has a lower plasma frequency compared to aluminum. Thus the values of effective plasma frequency of copper photonic crystals are also lower than those for the aluminum photonic crystals. So, the effective plasma frequency is in proportion to the plasma frequency of metals for each filling fraction.

TABLE (1). Parameters and root mean square of correlation for the effective plasma frequency equation in all filling fractions (copper).

Filling Fraction	*a*	*b*	R_{rms}
0.1	0.2809	1.3670	0.9291
0.2	0.4051	0.1706	0.9546
0.3	0.5133	0.1983	0.9589
0.4	0.6241	0.2392	0.9489
0.5	0.7570	0.2752	0.9643
0.6	0.8910	0.3133	0.9650
0.7	1.0320	0.3516	0.9638
0.8	1.1950	0.3999	0.9649
0.9	1.5280	0.5405	0.9727

TABLE (2). Parameters and root mean square of correlation for the effective plasma frequency equation in all filling fractions (aluminum).

Filling Fraction	*a*	*b*	R_{rms}
0.1	0.3822	0.0939	0.9610
0.2	0.5263	0.1125	0.9822
0.3	0.6531	0.1275	0.9840
0.4	0.7911	0.1488	0.9815
0.5	0.9483	0.1754	0.9808
0.6	1.114	0.201	0.9803
0.7	1.292	0.2279	0.9795
0.8	1.507	0.2662	0.9801
0.9	2.092	0.4466	0.9668

Case II

The investigation is expanded to photonic crystals in metallic media. The effective plasma values are obtained from the band structure graphs using equation 3. Figures 7 and 8 show the relationship between the effective plasma frequency and relative permittivity for square lattices whereas Figures 9 and 10 show the same relationship for triangular lattices. The values of effective plasma frequency are higher when a high plasma frequency is used. But in Figure 8 where the filling fraction at 0.9, the effective plasma frequency is almost flat for all the dielectric rods used and at very low frequency. This phenomenon is not found in triangular lattices. So, during the derivation, we ignored this graph, because the value is constant and we do not need any equation to define it.

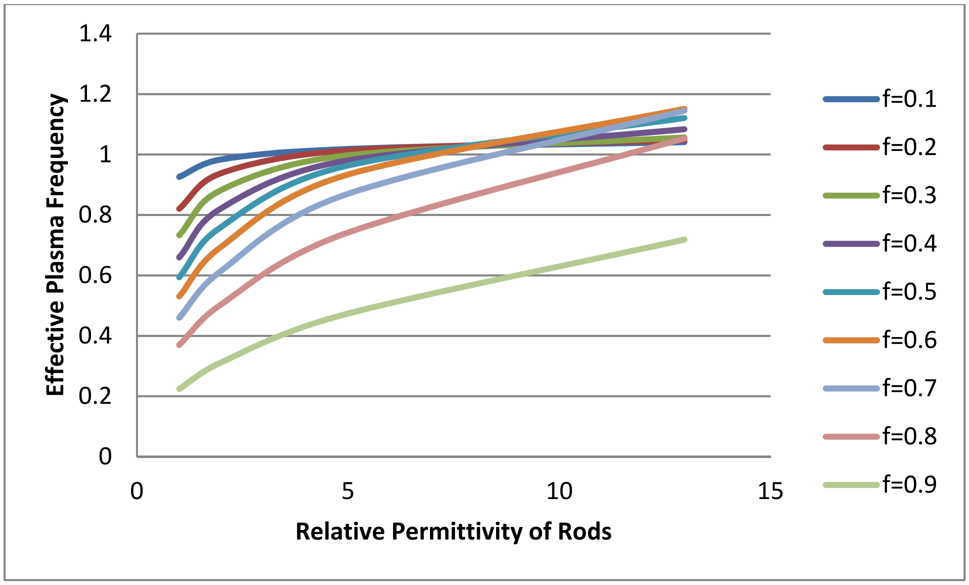

FIGURE 7. Effective plasma frequency versus relative permittivity for photonic crystals in copper media for square lattice arrangement for various filling fractions.

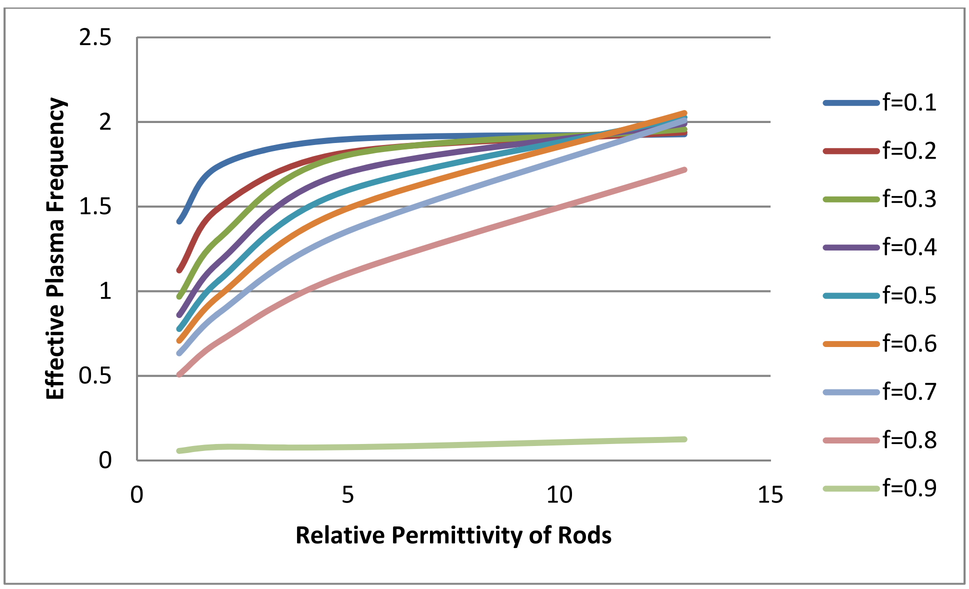

FIGURE 8. Effective plasma frequency versus relative permittivity for photonic crystals in aluminum media for square lattice arrangement for various filling fractions.

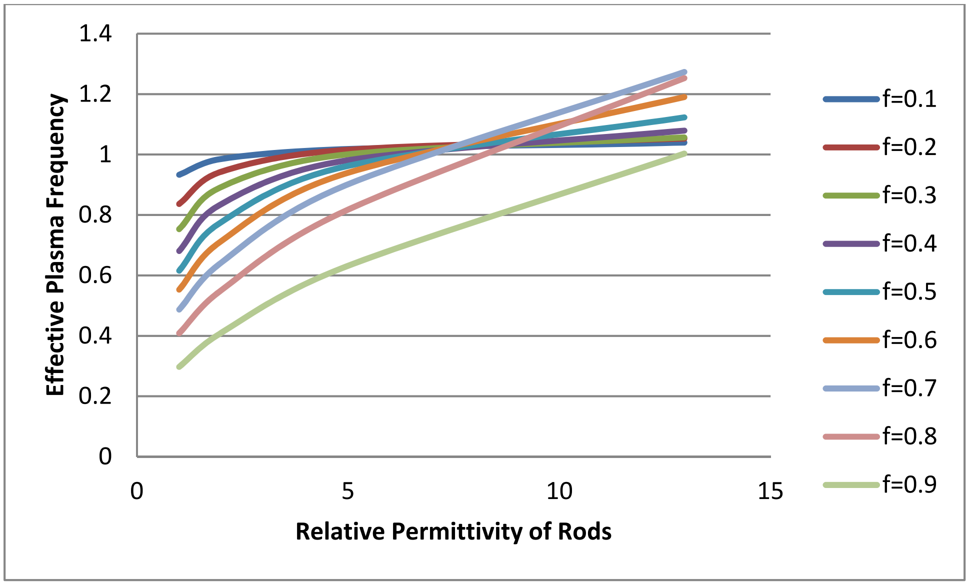

FIGURE 9. Effective plasma frequency versus relative permittivity for photonic crystals in copper media for triangular lattice arrangement for various filling fractions.

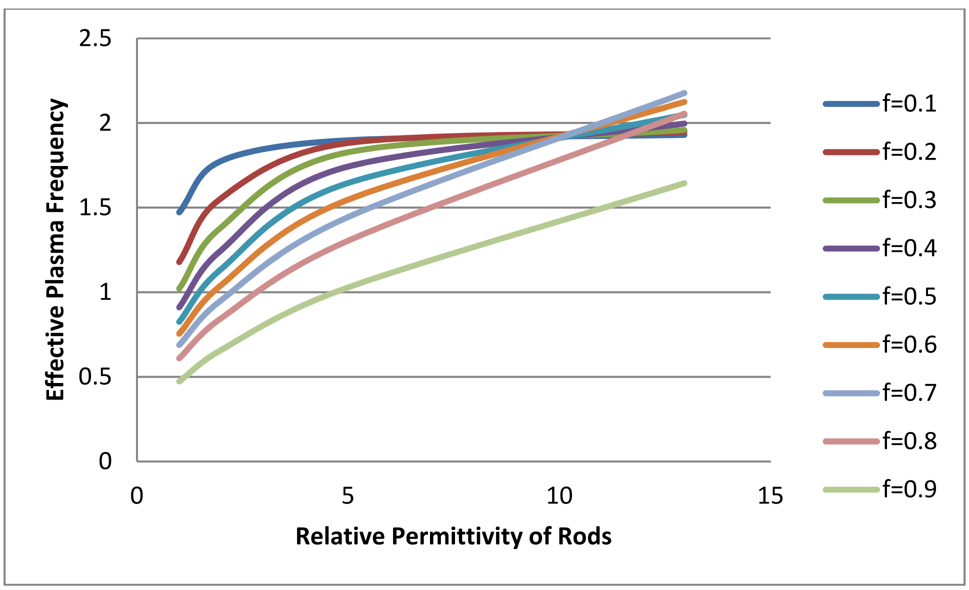

FIGURE 10. Effective plasma frequency versus relative permittivity for photonic crystals in aluminum media for triangular lattice arrangement for various filling fractions.

Figures 7 and 9 show similar values for both square and triangular arrangements, as do Figures 8 and 10, if the filling fraction 0.9 in Figure 8 ignored. As shown in the previous section, photonic crystals in metallic media also exhibit the same characteristic, namely that the lattice arrangement does not affect the effective plasma frequency. But, the graph patterns are different from those in the previous section. The effective plasma frequency will increase when the relative permittivity increases. When a higher value of relative permittivity is used, the effective plasma frequency is higher than the original plasma frequency. For a higher plasma frequency metal, namely aluminum, the effective plasma frequency rises even higher. But all the filling fractions will match the original plasma frequency when the relative permittivity is from 7 to 9 for copper, whereas it will rise above 11 for aluminum. The meeting point is actually the normalized plasma frequency for each metal: copper ($\omega_p a/2\pi c = 1.015$) and aluminum ($\omega_p a/2\pi c = 1.8939$). This brings up a characteristic of metal photonic crystals: even when mixed with dielectric rods, they still behave as metals for certain values of the relative permittivity. So, an equation between the effective plasma frequency and relative permittivity is obtained by using regression method in SigmaPlot® and is shown as below. The equation is a hyperbolic equation.

$$\upsilon_{\text{eff}} = \frac{a\varepsilon}{b+\varepsilon} \tag{6}$$

where *a* and *b* is a constant parameters. This numerical model produces a high correlation between the analytical result and statistical data. Table 3 shows the variation of the parameters *a* and *b* when the filling fraction changes. This means that the parameters depend on the physical dimensions of the photonic crystals, namely the radii of the rods and the lattice constant. Another property that we need consider is the relationship between plasma frequency and effective plasma frequency. When a higher plasma frequency is used, the value of the effective plasma frequency increases. So, the plasma frequency is proportional to the effective plasma frequency.

TABLE (3). Parameters and root mean square of correlation for the effective plasma frequency equation in all filling fractions (copper).

Filling Fraction	a	b	R_{rms}
0.1	1.0490	0.1334	0.9979
0.2	1.0750	0.3054	0.9983
0.3	1.0970	0.4934	0.9998
0.4	1.1370	0.7439	0.9976
0.5	1.1960	1.0840	0.9921
0.6	1.2620	1.5370	0.9869
0.7	1.3020	2.1260	0.9827
0.8	1.2590	2.9280	0.9786
0.9	0.9098	3.8580	0.9775

Table (4). Parameters and root mean square of correlation for the effective plasma frequency equation in all filling fractions (aluminum).

Filling Fraction	*a*	*b*	R_{rms}
0.1	2.027	0.3966	0.9595
0.2	2.090	0.8193	0.9936
0.3	2.180	1.238	0.9915
0.4	2.257	1.698	0.9972
0.5	2.357	2.251	0.9948
0.6	2.472	2.928	0.9883
0.7	2.529	3.712	0.9818
0.8	2.249	4.390	0.9790

Two simple numerical models are suggested in equation 5 and 6, for photonic crystals containing metallic components and photonic crystals in metallic media respectively. But, these two equations include the relative permittivity. The models are different from those in the literature. The filling fraction or radius of the rods will affect only the constants *a* and *b* because different filling fractions will give different values of effective plasma frequency but the same graph pattern. Another behavior of the effective plasma frequency is that the lattice arrangement does not affect its value. It produces a similar value but not exactly for certain filling fractions although in different lattice arrangements. These numerical models are not perfect, as they still suffer from some uncertainties. But at least they can explain the effective plasma frequency characteristic of photonic crystals when data for both frequency-independent materials and frequency-dependent materials are considered.

IV. CONCLUSIONS

We have successfully extracted the values of effective plasma frequency of photonic crystals containing metallic components and photonic metallic media from $\varepsilon = 1$ to $\varepsilon = 12.96$. Using these values, we successfully developed two numerical models that correlate the effective plasma frequency and the relative permittivity. The effective plasma frequency for square lattices and triangular lattices are the same. Thus the effective plasma frequency is not affected by the lattice arrangement. The effective plasma frequency is proportional to the plasma frequency for both Case I and Case II. For Case II, the effective plasma frequency will be close to the plasma frequency for relative permittivity from around 7 to 10. This is not found for Case I. But there are several parameters affecting the effective plasma frequency, including filling fraction, lattice constant, and relative permittivity.

1. E. Yablonovitch, *Physical Review Letter,* **58**, 2059-2062 (1987).
2. S. John, *Physical Review Letter,* **58**, 2486-2489 (1987).
3. J. B. Pendry, *Physical Review B* **85**, 3966-3969 (2000).
4. S. Brand, R. A. Abram, and M. A. Kaliteevski, *Physical Review B* **75**, 035102(1)-035102(7) (2007).
5. X. Xu, Y. Xi, D. Han, X. Liu, J. Zi, and Z. Zhu, *Applied Physics Letter,* **86**, p. 3, (2005).
6. J. B. Pendry, *Physical Review Letters* **76**, p. 4, (1996).
7. T. Nikolajsen, K. Leosson, and S. I. Bozhevolnyi, *Applied Physics Letters* **85**, 5833-5835, (2004).
8. R. Ortuno, C. Garcia-Meca, F. J. Rodriguez-Fortuno, J. Marti, and A. Martinez, *Physical Review B* **79**, 075425(1)-075425(10), (2009).
9. J. B. Pendry, A. J. Holdenz, D. J. Robbinsz, and W. J. Stewartz, *Journal of Physics: Condensed Matter* **10**, 4785-4809, (1998).
10. A. K. Sarychev and V. M. Salaev, *Condensed Matter,* 2001.
11. S. I. Maslovski, S. A. Tretyakov, and P. A. Belov, *Microwave and Optical Technology Letters* **35**,47-51 (2002).
12. S. Tretyakov, *Analytical Modeling in Applied Electromagnetics*: Artech House Publishing, 2004.
13. V. Kuzmiak and A. A. Maradudin, " *Physical Review B* **58**, 7230-7251 (1998).
14. K. L. Low, M. Z. M. Jafri, and S. A. Khan, *Progress In Electromagnetic Research M* **12**, 67-79 (2010).
15. K. L. Low, M. Z. M. Jafri, and S. A. Khan, *Chinese Journal of Physics* **47**,853-861, (2009).