Lattice Models on Cayley Tree with Competing Interactions

Nasir Ganikhodjaev[a] and Mohd Hirzie Bin Mohd Rodzhan[a]

[a]*International Islamic University Malaysia, Faculty of Science, Jalan Sultan Ahmad Shah, Bandar Indera Mahkota, 25200 Kuantan, Pahang, Malaysia*

(Received: 15.1.2018 ; Published: 7.6.2018)

Abstract. The Ising models with few competitive interactions play a significant role in studying the behavior of spin glasses. In this paper we review some generally known results about models on Cayley tree and present new ones. It is known that a phase diagram of models with competitive interbranch neighbors on Cayley tree of second order can contain ferromagnetic, paramagnetic, antiferromagnetic phases and paramodulated phase with period 2 only. We investigate the Uncle-Nephew Ising model on Cayley tree of third order with competitive interbranch neighbors and show that it has very rich phase diagram with many modulated phases, consisting of periodic (commensurate) and aperiodic (incommensurate) domains.

Keywords: Ising Model, Cayley tree, Phase diagram, Modulated Phase.

I. INTRODUCTION

The Ising model was created by Wilhelm Lenz in 1920, but it is named after Ernst Ising, a student of Lenz. The model was the subject of his doctoral dissertation in 1925 (see [1,2] for a history of the subject). He aimed to clarify certain empirically observed facts about ferromagnetic materials. Discrete random variables represent magnetic dipole moments of atomic spins with two states (+1 or -1). The spins are arranged in a countable graph and it is permitted for each spin to interact with its nearest neighbors.

The Ising model on a Cayley tree has been discussed by many authors [3-10]. Note that the investigation of lattice models on trees is reduced to the analysis of nonlinear recursion equations generated by the model. Thus the phase diagram of such models is fully determined by attractors of corresponding nonlinear recursion equations. Note that Ising model with nearest neighbor interactions only is not disarranged [3] because the Cayley tree is the graph without loops and multiple edges. In this case its nonlinear recursion equations have only fixed points as attractor, corresponding to paramagnetic and ferromagnetic phases. When the nearest-neighbor coupling constant changes sign, these phases bifurcated to paramodulated phase with period 2 and antiferromagnetic phase [3-5].

For Ising model with intrabranch competing interactions frustration effects are possible [3-5].

In this paper we review some well-known results about phase diagram of models on Cayley tree of second order with competing interbranch and intrabranch interactions. Then we present new results for Uncle-Nephew Ising model on Cayley tree of third order.

The plan of the paper as follows. In the second section, we define the models on the Cayley tree of second order and the relevant dynamical systems. We display the relevant field-temperature phase diagrams also in this section. In the third section, we discuss the Uncle-Nephew Ising model on the Cayley tree of third order and display its phase diagrams. Finally we summarize our findings in the last section.

II. MODELS ON CAYLEY TREE OF SECOND ORDER AND THEIR PHASE DIAGRAM

In this paper we consider the models on a Cayley tree of second (third) order, i.e. a graph without cycles, such that exactly 3 (respectively, 4) edges originate from each vertex. We select two types of neighbors (bonds): intrabranch and interbranch (see Fig.1). Two vertices belonging to the same branch of the tree are called *intrabranch neighbors* and two vertices belonging to the different branches of the tree are called *interbranch neighbors*.

We define our models by considering that each site of the Cayley tree is assigned with a ½ spin variable which is coupled by exchange interactions with its nearest neighbors (coupling constant J_1), intrabranch (interbranch) second-nearest neighbors (coupling constant $J_{2\mathrm{p}}$ (respectively $J_{2\mathrm{o}}$)) and intrabranch third-nearest neighbors (coupling constant $J_{3,p}$). Below we consider two kinds of interbranch third-nearest neighbors: left and right (with coupling constants $J_{3,l}$ and $J_{3,r}$ respectively). The interbranch third-nearest neighbors $<x,y>$ is called left (right) if x belongs (N-1)- generation and y belongs (N)-generation (respectively, x belongs (N)-generation and y belongs (N-1)-generation) for some N (see Fig. 1).

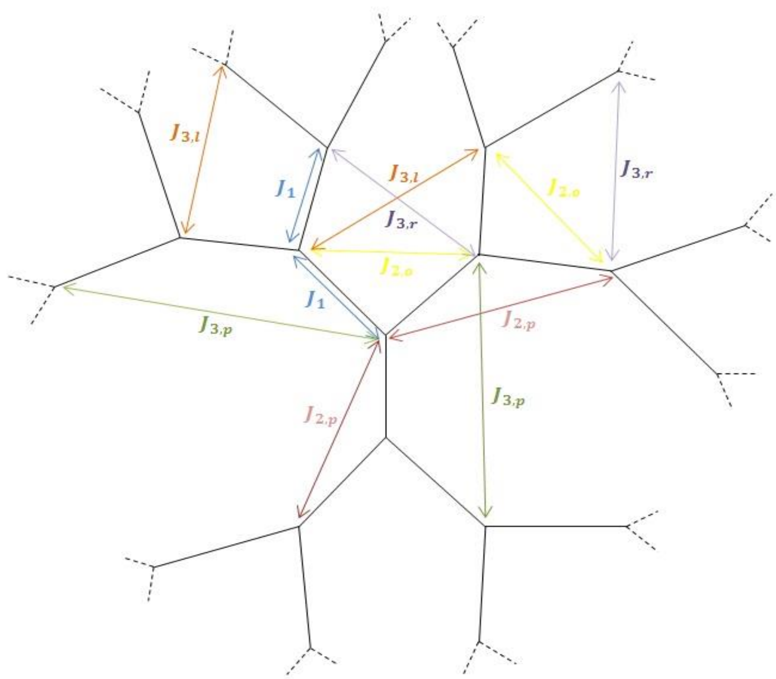


FIGURE 1. Portion of a Cayley tree of second order. Blue lines denote nearest neighbor coupling; red (yellow) lines denote intrabranch (respectively, interbranch) second nearest neighbor coupling; green (brown and purple) lines denote inrabranch (respectively, left and right interbranch) third nearest neighbor coupling.

For all models considered below we produce the recurrent equations using approach suggested by Vannimenus, namely, approximation of an infinite set by a sequence of its finite subsets [3].

Let V be a set of vertices of considered Cayley tree and $\{V_n\}$ be an increasing sequence of finite subsets of V such that $\bigcup V_n = V$. Then we produce recurrence equations relating the partition function on V_n to the partition function on its subset V_{n-1}. Assume that the initial conditions for the first subset V_1 are given. Then, the limit behavior of the recurrence equations indicate how their influence propagates the whole tree.

Below we consider the following models:

Ising model with competing interbranch next-nearest neighbors

The Hamiltonian for this model can be written as:

$$H(\sigma) = -\sum_{<x,y>} J_1 \sigma(x)\sigma(y) - \sum_{\underline{>x,y<}} J_{2,o} \sigma(x)\sigma(y) \qquad (1)$$

where the first summation involves all pairs of nearest neighbors and second summation is restricted to pairs of sites belonging to the different branches as explained above. This model was considered in [4], where the authors produced the following recurrence equation:

$$u_n = \frac{\theta^2\theta_1^2 u_{n-1}^2 + 2\theta u_{n-1} + \theta^2}{\theta^2\theta_1^2 + 2\theta_1 u_{n-1} + \theta^2 u_{n-1}^2}, \qquad n = 2,3,... \qquad (2)$$

where $\theta = \exp(\beta J_1)$ and $\theta_1 = \exp(J_{2,0})$. One iterates this recurrence equation (2) and observes its behavior after a large number of iterations. The resultant phase diagram with respect to ordinate T/J_1 and abscissa $-J_{2,o}/J_1$ is shown in Fig. 2a. It consists of well-known paramagnetic (P), ferromagnetic (F) and antiferromagnetic (AF) phases only. Thus competing intrabranch interaction does not influence to phase diagram of original Ising model.

Ising model with competing intrabranch next-nearest neighbors

The Hamiltonian of this model is defined as follows:

$$H(\sigma) = -\sum_{<x,y>} J_1 \sigma(x)\sigma(y) - \sum_{\underline{\underline{>x,y<}}} J_{2,p} \sigma(x)\sigma(y) \qquad (3)$$

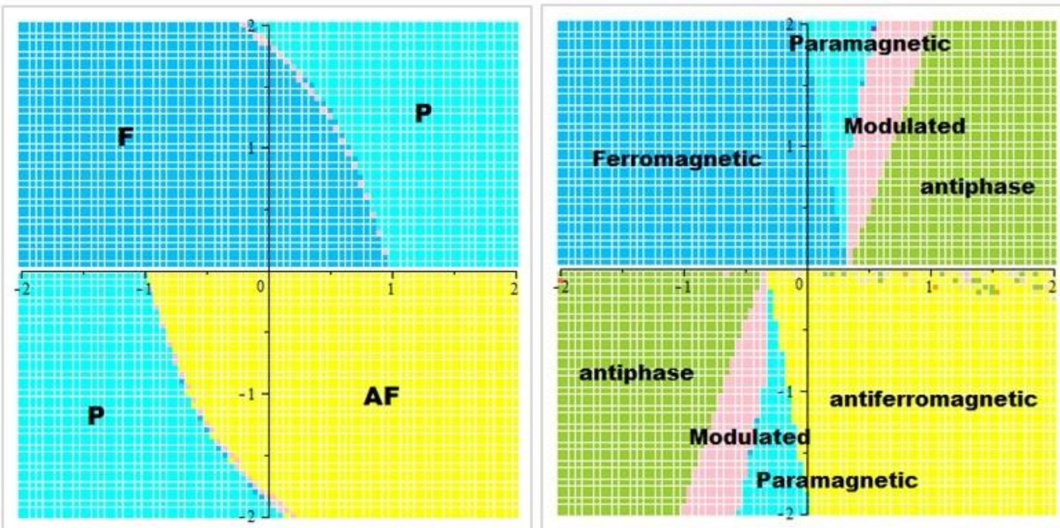
where the sum in the first term ranges over all nearest-neighbors and the second sum ranges over all interbranch next-nearest neighbors. As shown by Vannimenus [3], who considered this model, it can be solved in terms of recursion relations for the magnetization per spin in successive generations of the tree. He produced the following relations:

$$x' = \frac{1}{a^2 D}[(1+b^2x)^2 + (y_1 + b^2 y_2)^2],$$

$$y_1' = \frac{2}{D}(b^2 y_1 + y_2)(b^2 + x) \quad \text{and} \quad y_2' = -\frac{2}{a^2 D}(y_1 + b^2 y_2)(1 + b^2 x) \qquad (4)$$

with $D=(b^2+x)(b^2 y_1+y_2)$, where $a=\exp(\beta J_1)$ and $b=\exp(\beta J_{2,p})$.

Starting from random initial conditions, one iterates these recurrence relations (4) and observes their behavior after a large number of iterations. If its attractor is a fixed point (x^*, y_1^*, y_2^*), then it corresponds to a paramagnetic phase when $y_1^*=y_2^*=0$, and corresponds to a ferromagnetic phase when $y_1^*, y_2^* \neq 0$. The finite attractor corresponds to periodic (commensurate) phase and infinite attractor corresponds to aperiodic (incommensurate) phase.



(a) (b)

FIGURE 2. (a) Phase diagram of the model with intrabranch competing interactions; (b) Phase diagram of the Vannimenus model.

The resultant phase diagram is shown in the Fig.2b. It contains new modulated phase, in addition to the expected paramagnetic (P) and ferromagnetic (F) ones. Thus the considered model has a very rich phase diagram with many modulated phases. Here *antiphase* is antiferromagnetic phase with (+ + - -) periodicity and will be denoted <2> for compactness.

Ising model with competing intrabranch and interbranch next-nearest neighbors

The Hamiltonian of this model is defined as follows:

$$H(\sigma) = -\sum_{<x,y>} J_1\sigma(x)\sigma(y) - \sum_{\underline{>x,y<}} J_{2,o}\sigma(x)\sigma(y) - \sum_{\underline{\underline{>x,y<}}} J_{2,p}\sigma(x)\sigma(y) \quad (5)$$

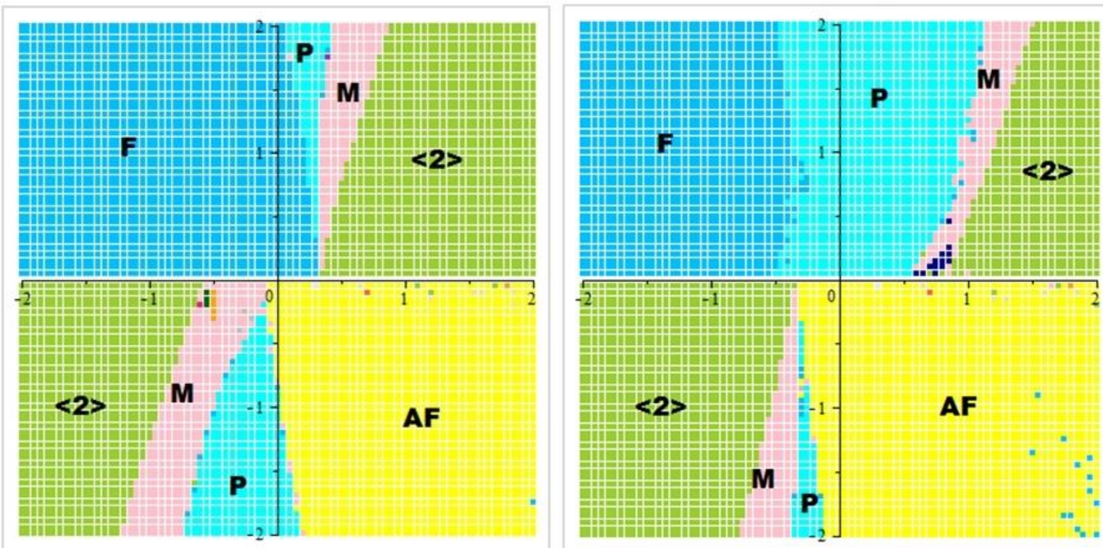
This model, considered by Mariz *et al*. [5], is an extension of previous model with the following recursion relations:

$$x' = \frac{1}{a^2 D}[b^4(x^2+y_2^2)+2(b/c)^2(x+y_1 y_2)+(1+y_1^2)],$$

$$y_1' = \frac{2}{D}[b^4 y_1+(b/c)^2(y_2+y_1 x)+y_2 x] \quad \text{and} \quad y_2' = -\frac{2}{a^2 D}[b^4 y_2 x+(b/c)^2(y_2+y_1 x)+y_1] \quad (6)$$

with $D = b^4(1+y_1^2) + 2(b/c)^2(x + y_1 y_2) + (x^2 + y_2^2)$, where $a = \exp(\beta J_1)$, $b = \exp(\beta J_{2,p})$, and $c = \exp(\beta J_{2,0})$.

These equations (6) generalize previous equations (4), which are recovered for c = 1. The resultant phase diagrams are shown in the Fig.3. The effect of the $J_{2,o}$ interaction one can see comparing Fig.3a and Fig.3b.



(a) (b)

FIGURE 3. Phase diagram of the Mariz et al model: (a) $-J_{2,o}/J_1$ = -0.725; (b) $-J_{2,o}/J_1$ =1.525.

Ising model with competing intrabranch next-nearest and third-nearest neighbors

The Hamiltonian for this model can be written as:

$$H(\sigma) = -\sum_{<x,y>} J_1 \sigma(x)\sigma(y) - \sum_{>x,y<} J_{2,p} \sigma(x)\sigma(y) - \sum_{<x,y>} J_{3,p} \sigma(x)\sigma(y) \qquad (7)$$

where the third sum ranges over all intrarbranch third-nearest neighbors. This model is the generalization of the Vannimenus [3] model, which was firstly considered by da Silva and Coutinho [6]. In [6] the authors have generalized the approach used by Thompson [7]. In [8] we investigated the model producing the following recurrent equations:.

$$x_1' = b^{-2} D^{-1}[(cx_2 + c^{-1}x_3)^2 + (cy_3 + c^{-1}y_4)^2], \qquad x_2' = a^2 D^{-1}[(c^{-1}x_2 + cx_3)^2 + (c^{-1}y_3 + cy_4)^2],$$

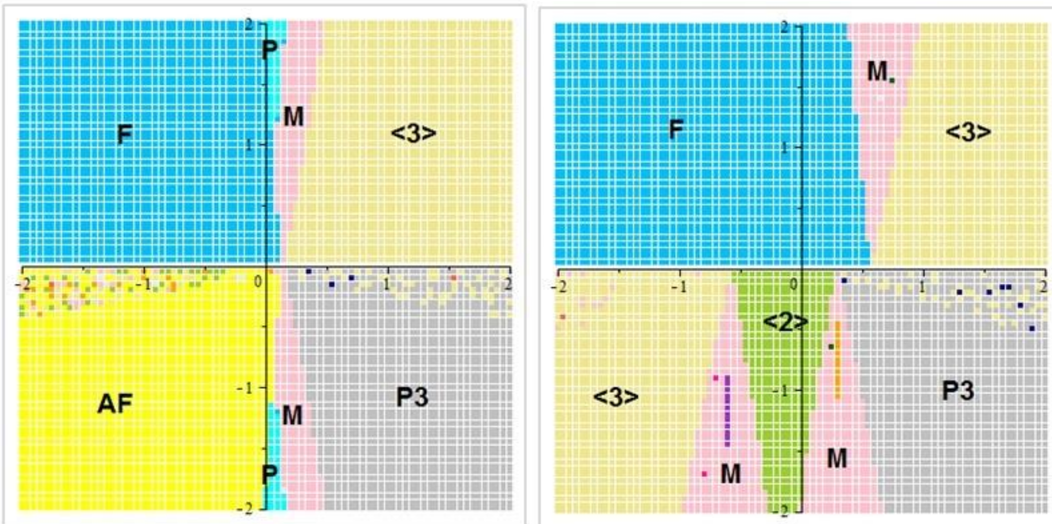
$$x_1' = a^2 b^2 D^{-1}[(cx_1 + c^{-1})^2 + (c^{-1}y_1 + cy_2)^2],$$

$$y_1' = 2D^{-1}[y_1(c^2 + x_1) + y_2(c^{-2}x_1 + 1)], \qquad y_2' = 2D^{-1}[y_3(c^2 x_2 + x_3) + y_4(c^{-2}x_3 + x_2)],$$

$$y_3' = -2D^{-1}[y_3(c^{-2}x_2 + x_3) + y_4(c^2 x_3 + x_2)], \qquad y_4 = -2D^{-1}[y_1(c^{-2} + x_1) + y_2(c^2 x_1 + 1)] \qquad (8)$$

where $a = \exp(\beta J_1)$, $b = \exp(\beta J_{2,p})$, $c = \exp(\beta J_{3,p})$ and $D = (c + c^{-1}x_1)^2 + (cy_1 + c^{-1}y_2)^2$.

As before we iterate these recurrence relations and display corresponding phase diagram. iterations. The resultant phase diagram is shown in Fig.4. One can see the appearance of a periodic (+ + + - - -) antiphase <3> in addition to the previous ones and also phase with period 3 (P3).



(a) (b)

FIGURE 4. Phase diagram of the da Silva and Coutinho model: (a)$J_{2,p} = 0$; (b) $-J_{2,p}/J_1 = 1$.

Ising model with competing interbranch third-nearest neighbors

The Hamiltonian of this model is defined as follows:

$$H(\sigma) = -\sum_{<x,y>} J_1 \sigma(x)\sigma(y) - \sum J_{3,r}\sigma(x)\sigma(y) - \sum J_{3,l}\sigma(x)\sigma(y) \qquad (9)$$

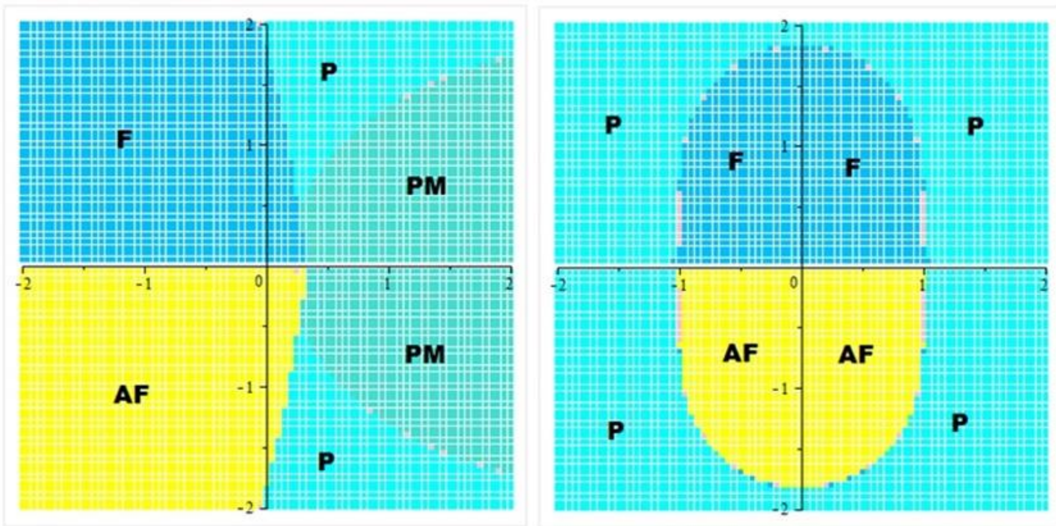
where the second and third sums range over all interbranch third-nearest neighbors.

Considering a Cayley tree as a genealogical or family tree, we can call this model as the Uncle-Nephew Ising model [9]. The particular case of this model with $J_{3,r} = J_{3,l} = J_3$ was considered in [9, 10] and produced the following recurrent equations:

$$\begin{aligned} x' &= aD^{-1}[(1 + a^{-2}b^2 + 2bx)^2 - (1 - a^{-2}b^2)^2 y^2], \\ y' &= 2D^{-1}(b^2 - a^{-2})[2bx + b^2 + a^{-2}]y \end{aligned} \qquad (10)$$

where $a = \exp(2\beta J_1), b = \exp(2\beta J_3)$ and $D = (2bx + a^{-2} + b^2)^2 + (b^2 - a^{-2})^2 y^2$.

The resultant phase diagram is shown in Fig.5a. It contains a new paramodulated (PM) phase in addition to the expected paramagnetic (P), ferromagnetic (F) and antiferromagnetic (AF) ones. Here, the paramodulated phase is a phase with period 2 and zero average magnetization.



(a) (b)

FIGURE 5. Phase diagram of the Uncle-Nephew Ising model: (a) $J_{3,r}/J_{3,l}$=1; (b) $J_{3,r}/J_{3,l}$ = -1.

In addition, one can produce the recurrent equations for general case $J_{3,r} \neq J_{3,l}$ and plot its corresponding phase diagram. For example in the case $J_{3,r} = -J_{3,l}$, the resultant phase diagram is shown in Fig.5b. It contains the well-known paramagnetic (P), ferromagnetic (F) and antiferromagnetic (AF) phases only.

III. UNCLE-NEPHEW ISING MODEL ON CAYLEY TREE OF THIRD ORDER AND ITS PHASE DIAGRAM

In this section we consider Uncle-Nephew Ising model on a Cayley tree of third order with the following Hamiltonian:

$$H(\sigma) = -\sum_{<x,y>} J_1\sigma(x)\sigma(y) - \sum J_{3,r}\sigma(x)\sigma(y) - \sum J_{3,l}\sigma(x)\sigma(y) \qquad (11)$$

where the second and third sums range over all interbranch third-nearest neighbors on a Cayley tree of third order.

In this paper we consider the particular case $J_{3,r} = J_{3,l} = J_3$ and as above assume:

$$a = \exp(2\beta J_1) \quad \text{and} \quad b = \exp(2\beta J_3).$$

In this case the corresponding dynamical is essentially complicated then (10).

Let

$$\begin{aligned} D &= 27a^{12}b^{12}(a^2b^4+1)^3x^3 + 27a^8b^8(a^6b^{12}+1)(a^2b^4+1)^2x^2 + 9a^4b^4(a^2b^4+1)(a^6b^{12}+1)^2x \\ &+ 9a^4b^4(a^2b^4+1)(a^6b^{12}-1)^2xy_1^2 + 81a^{12}b^{12}(a^2b^4-1)(a^4b^8-1)xy_2^2 \\ &+ 54a^8b^8(a^6b^{12}-1)(a^4b^8-1)xy_1y_2 + 54a^8b^8(a^6b^{12}-1)(a^4b^8-1)y_1^2 \\ &+ 27a^8b^8(a^6b^{12}+1)(a^2b^4-1)^2y_2^2 + 18a^4b^4(a^{12}b^{24}-1)(a^6b^{12}-1)y_1y_2 + (a^6b^{12}+1)^3. \end{aligned}$$

Then the corresponding recurrence equations have the following form:

$$
\begin{aligned}
x' &= a^{-2}b^{12}D^{-1}\{27a^{12}b^{4}(1+a^{2})^{2}(a^{2}+b^{4})x^{3} \\
&+9a^{8}(1+a^{2})^{2}(a^{2}+b^{4})(2a^{4}b^{4}+a^{4}-3a^{2}b^{4}+b^{8}+2b^{4})x^{2} \\
&+3a^{4}(a^{6}+1)(a^{2}+1)(a^{2}+b^{4})[a^{4}(b^{4}+2)-3a^{2}b^{4}+2b^{8}+b^{4}]x \\
&+3a^{4}(a^{6}-1)[a^{8}(b^{4}-2)+a^{6}(b^{8}-2)+a^{2}(2b^{12}-b^{4})+2b^{12}-b^{8}]xy_1^{2} \\
&-18a^{8}(a^{2}-1)[a^{8}+a^{6}(2b^{4}-2b^{8}+1)+2a^{4}(b^{4}-b^{8})+a^{2}(2b^{4}-2b^{8}-b^{12})-b^{12}]xy_1y_2 \\
&-27a^{12}b^{4}(a^{2}-1)[a^{4}-3a^{2}(b^{4}-1)-b^{4}]xy_2^{2}+(a^{6}-1)[a^{12}-3a^{6}(b^{12}-1)-b^{12}]y_1^{2} \\
&-6a^{4}(a^{2}-1)[a^{12}b^{4}+(a^{10}+a^{8}+a^{4}+a^{2})(b^{4}-b^{8})+a^{6}(2-2b^{12}-b^{8}+b^{4})-b^{8}]y_1y_2 \\
&-9a^{8}(a^{2}-1)[a^{8}(2b^{4}-1)+a^{6}(1-2b^{8})+a^{2}(2b^{4}-b^{12})-2b^{8}+b^{12}]y_2^{2} \\
&+(a^{6}+1)^{2}(a^{6}+b^{12})\}
\end{aligned}
$$

$$
\begin{aligned}
y_1' &= D^{-1}\{(a^{6}b^{12}-1)^{3}y_1^{3}+27a^{12}b^{12}(a^{2}b^{4}-1)^{3}y_2^{3}+27a^{4}b^{4}(a^{6}b^{12}-1)^{2}(a^{2}b^{4}-1)y_1^{2}y_2 \\
&+27a^{8}b^{8}(a^{6}b^{12}-1)^{2}(a^{2}b^{4}-1)^{2}y_1y_2^{2}+27a^{8}b^{8}(a^{6}b^{12}-1)^{2}(a^{2}b^{4}+1)^{2}x^{2}y_1 \\
&+81a^{12}b^{12}(a^{4}b^{8}-1)^{2}x^{2}y_2+18a^{4}b^{4}(a^{6}b^{12}-1)(a^{8}b^{16}+a^{6}b^{12}+a^{2}b^{4}+1)xy_1 \\
&+54a^{8}b^{8}(a^{4}b^{8}-1)^{2}(1-a^{2}b^{4}+a^{4}b^{8})xy_2+3(a^{6}b^{12}-1)(a^{6}b^{12}+1)^{2}y_1 \\
&+9a^{4}b^{4}(a^{4}b^{8}-1)^{2}(1-a^{2}b^{4}+a^{4}b^{8})y_2\} \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
y_2' &= a^{-2}b^{12}D^{-1}\{-(a^{6}-b^{12})(a^{6}-1)^{2}y_1^{3}-27a^{4}b^{12}(a^{2}-1)^{2}(a^{2}-b^{4})y_2^{3} \\
&-3a^{4}(a^{6}-1)(a^{2}-1)(a^{2}-b^{4})[a^{4}(b^{4}+2)+3a^{2}b^{4}+2b^{8}+b^{4}]y_1^{2}y_2 \\
&+9a^{8}[-a^{10}(1+2b^{4})+2a^{8}(1+b^{4}+b^{8})-a^{6}(1+2b^{8})+a^{4}(2b^{4}+b^{12})-2a^{2}(b^{10}+b^{8}+b^{4}) \\
&+b^{12}-2b^{8}]y_1y_2^{2}+9a^{8}[a^{10}b^{4}+2a^{8}(b^{4}+b^{8}-1)+a^{6}(2b^{8}-1)+a^{4}(b^{12}-2b^{4}) \\
&+2a^{2}(b^{12}-b^{8}-b^{4})+b^{12}-2b^{8}-1)x^{2}y_1+27a^{12}(a^{2}+1)[a^{4}-3a^{2}(2b^{4}+1)-b^{4}]x^{2}y_2 \\
&+6a^{4}[a^{14}b^{4}+a^{12}b^{8}+2a^{8}(b^{12}-1)+2a^{6}(b^{12}-1)-a^{2}b^{4}-b^{8}]xy_1 \\
&+9a^{8}(a^{2}+1)[2a^{8}+2a^{6}(2b^{8}-2b^{4}-1)-4a^{4}b^{8}+2a^{2}((b^{12}+2b^{8}-2b^{4})-2b^{12}]xy_2 \\
&+a^{6}[a^{12}+3a^{6}b^{12}+2b^{12}-3]y_1+3a^{4}(a^{6}+)[a^{8}(2-b^{4})+a^{6}b^{8}+a^{2}(2b^{12}-b^{4})-2b^{12}+b^{8}]y_2\}.
\end{aligned}
$$

As before we iterate these recurrence relations (12) and display corresponding phase diagram. The resultant phase diagram is shown in Fig.6. In contrast to Fig.5, it contains new modulated (M) phases, in addition to the expected paramagnetic (P), ferromagnetic (F) and antiferromagnetic (AF) ones.

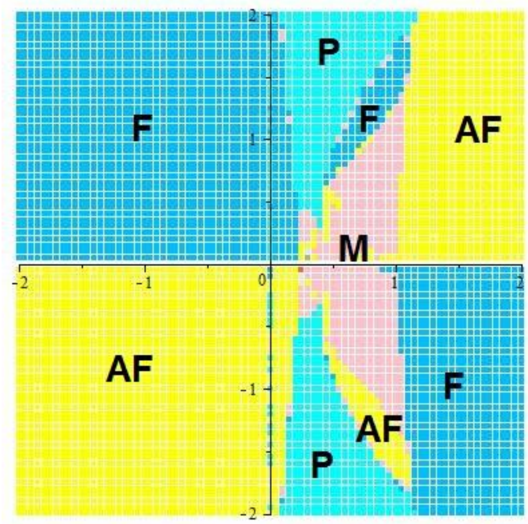


FIGURE 6. Phase diagram of the model of the Uncle-Nephew Ising model on the Cayley tree of third order.

IV. CONCLUSIONS

We have considered Ising models with competing next-nearest-neighbors and third-nearest-neighbors on Cayley tree of second and third order. It is shown that models with competing intrabranch neighbors on Cayley tree of second order have very rich phase diagram with many modulated phases whereas models with competing interbranch neighbors, in particular Uncle-Nephew Ising model, can have only antiferromagnetic (AF) and paramodulated (PM) phases with period 2, in addition to the expected paramagnetic (P) and ferromagnetic (F) phases. It is shown also that the Uncle-Nephew Ising model with interbranch third-nearest neighbors on Cayley tree of third order have very rich phase diagrams with many modulated phases.

ACKNOWLEDGMENTS

The work has been supported by the MOHE grant FRGS14-116-0357. Second author thanks International Islamic University Malaysia for providing financial support through research grant RIGS17-092-0667 and all facilities.

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