Killing Tensor of Five Dimensional Melvin's Spacetime

Ganesh Subramaniam[a,*], Hishamuddin Zainuddin[a,b] and Nurisya Mohd Shah[a,b]

[a]*Laboratory of Computational Sciences and Mathematical Physics, Institute for Mathematical Research, University Putra Malaysia (UPM), 43400 UPM Serdang, Selangor.*
[b]*Theoretical Physics Group, Department of Physics, Faculty of Science, University Putra Malaysia (UPM), 43400 UPM Serdang, Selangor.*
[*]*Email: ganesh_nesh@hotmail.my*

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Abstract. Killing tensors are generalizations of Killing vectors as objects that reflect the symmetries of spacetime. With recent interest in higher-dimensional spacetimes, construction of Killing tensors from lower dimensional ones may be useful. Our focus lies in the (4+1) dimensional Melvin's spacetime which describes a magnetic universe with a cylindrical symmetry. We constructed the Killing vectors and Killing tensors in 5-dimensional Melvin's spacetime. The Killing tensors are a linear combination of scalar times a metric and respective symmetric product of Killing vectors similar to those found by Garfinkle and Glass for the 4-dimensional case. It is relatively easy to write down Killing tensors of a particular spacetime admitting both commuting and hypersurface orthogonal Killing vectors.

Keywords: Killing vectors, Killing tensors, Melvin's spacetime.

I. INTRODUCTION

Today, research on higher dimensional spacetimes has gained vast attention from many researchers due to the developments of M-theory, braneworld cosmology, quantum gravity and etc. Symmetries are the topic that is mostly paid attention in these developments due to its close relation to the conservation laws. Particularly in the study of spacetime symmetries, Killing vectors are known to be an infinitesimal generator of isometries [1]. Any symmetric rank-2 Killing tensors X_{ab} (usually called Hidden Symmetries) which are the generalization of Killing vectors satisfy Killing equation $2X_{(ab;c)} = 0$, where a " ; " denotes the covariant derivatives with respect to the coordinates. Any symmetric rank-2 Killing tensors X_{ab} can be decomposed into $X_{ab} = A\xi_a\xi_b + B_{(a}\xi_{b)} + C_{ab}$ under the presence of hypersurface orthogonal Killing vector field ξ_a [2, 3]. B_a, C_{ab} are orthogonal to ξ_a and A is some constant.

Nowadays, Killing tensors are studied widely to understand the integrability of geodesic equation, separability of Hamilton-Jacobi equation, Klein Gordan equation and Dirac equation in both four and higher dimensional black hole spacetimes (refer to [4] for brief details) [5, 6, 7, 8]. Interestingly, studying about Killing tensors, Killing-Yano tensors, Conformal Killing tensors and Conformal Killing-Yano tensors enables researchers to study quantum mechanics in curved spacetime which relates general relativity with quantum mechanics [9, 10, 11, 12, 13].

Melvin's spacetime originated from the idea of Geons proposed by J. A. Wheeler in 1964 where he suggested that electromagnetic or gravitational wave can be held together under gravitational attraction contributed by their own energy [14, 15]. Later on, Geon was renamed as Magnetic or Melvin's Universe. In 1997, a thorough analysis of gravitational geons found by Hartle and Brill [16] were made and it was found that the geometry inside and outside regions of gravitational geons were similar to electromagnetic geons [17]. It was known that gravitational and electromagnetic geons are viable entities through stability analysis [18]. Recently, wormholes in Melvin's magnetic universe have also been studied [19]. The 4-dimensional magnetic universe is also studied in 5-dimensional Kaluza-Klein spacetime [20]. The five dimensional Melvin's metric considered in this work is closely related to the metric considered in [20] but the component of the metric in the fifth coordinate (ψ-direction) considered in this paper is more complicated than the metric considered in [20].

Our interest lies on the cylindrical Melvin's spacetime which describes Melvin's universe with axial symmetry [14, 15, 21]. Extensions from (3+1)D Melvin's spacetime to (4+1)D Melvin's spacetime may have interest in Kaluza-Klein or Brane-type solutions [22]. In [23], the authors introduced a new method to compute Killing tensors for (3+1)D Melvin's spacetime with symmetries. This method needs the hypersurface orthogonal Killing vector in each coordinate direction to compute Killing tensors. In this paper, we discussed the Killing tensors in (4+1)D Melvin's spacetime which is constructed by adding a coordinate to the (3+1)D Melvin's spacetime. Some of the problems in these methods will be discussed.

II. REVIEW OF THE METHOD AND RESULT IN (3+1)D

In [23], a method was introduced to find Killing tensors without computing Christoffel symbol. With this method, one starts with 2D Melvin's space to compute Killing tensor then uses the Killing tensor in 2D to compute Killing tensor in 3D Melvin's space and finally using the Killing tensor in 3D to find Killing tensor in (3+1)D Melvin's spacetime. The result found in [23] for (3+1)D Melvin's spacetime is

$$\begin{aligned} X'_{ab} =& c_1 g_{ab} + c_2 \gamma_a \gamma_b + c_3 \eta_a \eta_b + c_4 \eta_{(a} \gamma_{b)} + 2c_5 \gamma_{(a} \tau_{b)} + 2c_6 \eta_{(a} \tau_{b)} + 2c_7 \lambda_{(a} \eta_{b)} \\ &+ 2c_8 \lambda_{(a} \tau_{b)} + c_9 \gamma_{(a} \lambda_{b)} + c_{10} \lambda_a \lambda_b + c_{11} \tau_a \tau_b, \end{aligned} \tag{1}$$

where c with numbered subscripts are some constants, g_{ab} is the metric of the spacetime, λ_a, η_a, τ_a and γ_a are the Killing vector in the z, ϕ, t-directions and boost Killing vector (combination of Killing vector of z and t-directions). Take note that the expressions for the Killing tensor (1) only consist of the products of commuting Killing vectors only. This sparks the question of whether Killing tensors only consist of the sum of the product of only commuting Killing vectors. The fifth component (ψ - coordinate) of the metric (2) depends on the ϕ - coordinate which means that there is no commuting Killing vector (η_a) in the ϕ - direction [24]. Therefore in this paper, rather than finding Killing tensors in (4+1)D Melvin's metric, we also tries to show that Killing tensors do not necessarily consist of the sum of the product of commuting Killing vectors only. To show this, an ordinary method is used in this paper to compute Killing vector and construct a Killing tensor from the Killing vector found in (4+1)D Melvin's spacetime.

III. RESULTS AND DISCUSSION

The (4+1) dimensional Melvin's spacetime is given by the metric

$$ds^2 = a^2(-dt^2 + dr^2 + dz^2) + \left(\frac{r}{a}\right)^2 d\phi^2 + \left(\frac{r}{a}\sin[\phi]\right)^2 d\psi^2, \tag{2}$$

where $a = 1 + r^2$ (with r is radius). The last term was added using hyper-sphere coordinates. The last term added to the Melvin's metric in [23] is dependent on the ϕ - coordinate, for which there is no commuting Killing vectors in that direction [24]. Thus, it is not certain the existence of commuting Killing vectors term in equation (1) when using the Killing tensor (1) to compute Killing tensors in (4+1)D Melvin's spacetime as in [23].

In this paper, we first try to find Killing vector for the metric (2) and find parts of the Killing tensors by symmetrically multiplying Killing vectors. It is easily recognized that the metric (2) admits axial symmetry in z, t - directions and r, ϕ, ψ are the spherical extension of Euclidean space. Since it is difficult to directly solve Killing equations for five dimensional Melvin's metric (2), we use the Killing vectors in three dimensional Euclidean space whose metric (2) appeared embedded in a polar form. They are known to be [25]

$$\begin{aligned} &X_1 = \partial_x; && X_2 = \partial_y; && X_3 = \partial_z; \\ &X_4 = -y\partial_z + z\partial_y; && X_5 = -z\partial_x + x\partial_z; && X_6 = -x\partial_y + y\partial_x. \end{aligned} \tag{3}$$

To relate with metric (2), we notes that for spherical coordinates

$$x = r\sin[\phi]\cos[\psi]; \qquad y = r\sin[\phi]\sin[\psi]; \qquad z = r\cos[\phi], \tag{4}$$

equation (3) can be written as

$$X_1 = \sin[\phi]\cos[\psi]\partial_r + \frac{1}{r}\cos[\phi]\cos[\psi]\partial_\phi - \frac{1}{r}\frac{\sin[\psi]}{\sin[\phi]}\partial_\psi, \tag{5}$$

$$X_2 = \sin[\phi]\sin[\psi]\partial_r + \frac{1}{r}\cos[\phi]\sin[\psi]\partial_\phi + \frac{1}{r}\frac{\cos[\psi]}{\sin[\phi]}\partial_\psi, \tag{6}$$

$$X_3 = \cos[\phi]\partial_r - \frac{1}{r}\sin[\phi]\partial_\phi, \tag{7}$$

$$X_4 = \sin[\psi]\partial_\phi + \cot[\phi]\cos[\psi]\partial_\psi, \tag{8}$$

$$X_5 = -\cos[\psi]\partial_\phi + \cot[\phi]\sin[\psi]\partial_\psi, \tag{9}$$

$$X_6 = -\partial_\psi. \tag{10}$$

From equations (5) - (10), note that only equations (8) - (10) satisfy the Lie derivative of metric (2) (Killing equation)

$$\mathscr{L}_{X_i} g_{ab} = 0, \tag{11}$$

for $i = 1,2,3,4,5,6$. Therefore equations (8) - (10) are Killing vectors for the metric (2). Metric (2) admits another three Killing vectors which are part of the Killing vectors of (3+1)D Melvin's spacetime

$$X_7 = t\partial_z + z\partial_t, \tag{12}$$

$$X_8 = \partial_t, \tag{13}$$

$$X_9 = \partial_z. \tag{14}$$

Parts of Killing tensors can be symmetric products of Killing vectors with each other [26]. Therefore, a possible rank two Killing tensor for five dimensional Melvin's spacetime (2) is

$$\begin{aligned} KT =& \left(2\cot[\phi]\sin^2[\psi] - 2\cot[\phi]\cos^2[\psi] - 2\sin[\psi] + 2\cos[\psi]\right)\partial_{(\phi}\partial_{\psi)} \\ &+ \left(2t\sin[\psi] + 2\sin[\psi] - 2t\cos[\psi] - 2\cos[\psi]\right)\partial_{(\phi}\partial_{z)} \\ &+ \left(2z\sin[\psi] + 2\sin[\psi] - 2z\cos[\psi] - 2\cos[\psi]\right)\partial_{(\phi}\partial_{t)} \\ &+ \left(2t\cot[\phi]\cos[\psi] + 2\cot[\phi]\cos[\psi] + 2t\cot[\phi]\sin[\psi] + 2\cot[\phi]\sin[\psi]\right. \\ &\left. - 2t - 2\right)\partial_{(\psi}\partial_{z)} + \left(2z\cot[\phi]\cos[\psi] + 2\cot[\phi]\cos[\psi] + 2z\cot[\phi]\sin[\psi]\right. \\ &\left. + 2\cot[\phi]\sin[\psi] - 2z - 2\right)\partial_{(\psi}\partial_{t)} + \left(2z + 2t + 2\right)\partial_{(z}\partial_{t)} + \left(2\cot^2[\phi]\sin[\psi]\cos[\psi]\right. \\ &\left. - 2\cot[\phi]\cos[\psi] - 2\cot[\phi]\sin[\psi]\right)\partial_{(\psi}\partial_{\psi)} + 2z\partial_t\partial_t + 2t\partial_z\partial_z - 2\sin[\psi]\partial_\phi\partial_\phi. \end{aligned} \tag{15}$$

Lets substitute $\delta_a^z\partial_z = \lambda_a$, $\delta_a^\phi\partial_\phi = \eta_a$, $\delta_a^\psi\partial_\psi = \theta_a$ and $\delta_a^t\partial_t = \tau_a$ (with indices a and b

represent the coordinate directions) into equation (15), we obtain

$$
\begin{aligned}
KT =& \big(2\cot[\phi]\sin^2[\psi] - 2\cot[\phi]\cos^2[\psi] - 2\sin[\psi] + 2\cos[\psi]\big)\,\eta_{(a}\theta_{b)} \\
&+ \big(2t\sin[\psi] + 2\sin[\psi] - 2t\cos[\psi] - 2\cos[\psi]\big)\,\eta_{(a}\lambda_{b)} \\
&+ \big(2z\sin[\psi] + 2\sin[\psi] - 2z\cos[\psi] - 2\cos[\psi]\big)\,\eta_{(a}\tau_{b)} \\
&+ \big(2t\cot[\phi]\cos[\psi] + 2\cot[\phi]\cos[\psi] + 2t\cot[\phi]\sin[\psi] + 2\cot[\phi]\sin[\psi] \\
&- 2t - 2\big)\,\theta_{(a}\lambda_{b)} + \big(2z\cot[\phi]\cos[\psi] + 2\cot[\phi]\cos[\psi] + 2z\cot[\phi]\sin[\psi] \\
&+ 2\cot[\phi]\sin[\psi] - 2z - 2\big)\,\theta_{(a}\tau_{b)} + \big(2z + 2t + 2\big)\,\lambda_{(a}\tau_{b)} + \big(2\cot^2[\phi]\sin[\psi]\cos[\psi] \\
&- 2\cot[\phi]\cos[\psi] - 2\cot[\phi]\sin[\psi]\big)\,\theta_{(a}\theta_{b)} + 2z\tau_a\tau_b + 2t\lambda_a\lambda_b - 2\sin[\psi]\eta_a\eta_b.
\end{aligned}
\tag{16}
$$

Since the five dimensional metric is a rank two symmetric tensor, one can add the metric into equation (16) and the Killing tensor for five dimensional Melvin's spacetime can be expressed as

$$
\begin{aligned}
X_{ab} =& g_{ab} + \big(2\cot[\phi]\sin^2[\psi] - 2\cot[\phi]\cos^2[\psi] - 2\sin[\psi] + 2\cos[\psi]\big)\,\eta_{(a}\theta_{b)} \\
&+ \big(2t\sin[\psi] + 2\sin[\psi] - 2t\cos[\psi] - 2\cos[\psi]\big)\,\eta_{(a}\lambda_{b)} \\
&+ \big(2z\sin[\psi] + 2\sin[\psi] - 2z\cos[\psi] - 2\cos[\psi]\big)\,\eta_{(a}\tau_{b)} \\
&+ \big(2t\cot[\phi]\cos[\psi] + 2\cot[\phi]\cos[\psi] + 2t\cot[\phi]\sin[\psi] + 2\cot[\phi]\sin[\psi] \\
&- 2t - 2\big)\,\theta_{(a}\lambda_{b)} + \big(2z\cot[\phi]\cos[\psi] + 2\cot[\phi]\cos[\psi] + 2z\cot[\phi]\sin[\psi] \\
&+ 2\cot[\phi]\sin[\psi] - 2z - 2\big)\,\theta_{(a}\tau_{b)} + \big(2z + 2t + 2\big)\,\lambda_{(a}\tau_{b)} + \big(2\cot^2[\phi]\sin[\psi]\cos[\psi] \\
&- 2\cot[\phi]\cos[\psi] - 2\cot[\phi]\sin[\psi]\big)\,\theta_{(a}\theta_{b)} + 2z\tau_a\tau_b + 2t\lambda_a\lambda_b - 2\sin[\psi]\eta_a\eta_b.
\end{aligned}
\tag{17}
$$

IV. CONCLUSIONS

In this note, we have constructed the Killing vectors and Killing tensor for five dimensional Melvin's spacetime. While retaining the single spatial axial symmetry about z, the extension by 3-dimensional Euclidean space embedded in (4+1)D Melvin's spacetime using polar coordinate, allows us to compute Killing vectors without solving the full Killing equation for the (4+1)D Melvin's spacetime for which initially is very difficult to solve. The Killing tensor found here is a linear combination of metric and a constant times symmetric product of Killing vectors as found in [23] for four dimensional Melvin's spacetime. Coincidentally, the Killing tensors (trivial) for the (3+1)D Melvin's spacetime only consist of the product of commuting Killing vectors and all these Killing vectors are hypersurface orthogonal. Therefore, Killing tensors do not necessarily consist of the product of commuting Killing vectors only but must be hypersurface orthogonal as in the (4+1)D Melvin's spacetime under discussion. Furthermore, it is relatively easy to write down Killing tensors for a particular spacetime when all the Killing vectors in the spacetime are both commuting and hypersurface orthogonal Killing vectors. Here we conclude that one can still use the method introduced in [23] bearing in mind that the functional dependence of the metric on the coordinates may require recomputing the Killing vectors. The metric (2) we considered in these notes is closely similar to the metric considered in [20] except that the components of the metric considered in this paper is specific which might have some application in Kaluza-Klein theory and 5-dimensional electromagnetic related theories. It is expected that for even higher dimensional spacetimes, such complication will arise depending on what symmetries are assumed.

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