When Evolutionary Games and the Ising Model Meet

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Abstract. Evolutionary games with a self-questioning action updating mechanism have recently been mapped onto the Ising model. We discuss here an alternative viewpoint in making a connection between two fields of research that each has much to offer to the other.

Keywords: Evolutionary games, self-questioning adaptive mechanism, Ising model.

I. INTRODUCTION

The presentations in the sessions on complex systems in the Fifth International Meeting on Frontiers of Physics (IMFP 2017) held recently in Kuala Lumpur and organized by the Malaysian Institute of Physics and the Department of Physics at the University of Malaya indicated a good level of activities in this research area in the host country and the region. This short paper is a slight extension of the plenary talk given in the Meeting. Recently, Liu *et al.* [1] constructed an evolutionary game using a self-questioning adaptive mechanism that can be mapped on to the Ising model. As evolutionary games are crucial in studying the emergence of cooperation in a competing population [2] and the Ising model is one of the best studied models in statistical physics [3], a connection between them would allow the two fields to borrow ideas, techniques and results from each other. In Ref.[1], the mapping was established via arguments based on detailed balance. More on previous works on relating the games and opinion formation model to the Ising model [4,5,6,7,8] can be found in Ref.[1]. Here, we discuss the mapping from a slightly different point of view.

II. ISING MODEL

We start with the Ising model with nearest-neighbouring interaction described by

$$H = \frac{1}{2}\sum_{i,j\in\{i\}} J_{ij} s_i s_j - h\sum_i s_i = \sum_{<i,j>} J_{ij} s_i s_j - h\sum_i s_i \ , \qquad (1)$$

where J_{ij} signifies the interaction energy between neighboring spins and h is an external uniform field. The first term originates from the interaction of spins and the second term comes from an external field. The variable s_i takes on ±1 or simply ±. In the first form, the summations in the first term are over the neighbouring spins of spin i for every i. In the second form, the summation is over all distinct pairs of spins. In the standard point of view, $J_{ij} = J$ signifies the interaction energies of a pair of neighboring spins with $E_{++} = J, E_{+-} = -J, E_{-+} = -J, E_{--} = J$.

We could as well take a slightly different viewpoint. Let us assume that we know the *differences* $E_{+-} - E_{--} = -2J$ and $E_{-+} - E_{++} = -2J$, plus the condition that $E_{+-} = E_{-+}$. The difference $E_{+-} - E_{--} = -2J$, for example, can be interpreted as the change in the interaction energy in a pair if a spin, represented by the first index, makes a flip and it is a quantity often used in relation to the acceptance or rejection of an attempt on spin-flip in Monte Carlo simulations. The question is whether we could design J_{ij} in Equation (1) from the given information. To proceed, a referencing energy $-J$ can be set by $-J = E_{+-} = E_{-+}$. It follows from the required differences that $E_{++} = J$ and $E_{--} = J$, which are consistent with the standard point of view.

III. EVOLUTIONARY GAMES WITH SELF-QUESTIONING ADAPTIVE MECHANISM

The second viewpoint becomes useful when we consider the possible connection between evolutionary games and the Ising model. Obviously, similarities exist. While a spin can take on ±1, an agent in competing games can take on one of two strategies or actions: C or D, which may as well be represented by ±1. Depending on the contexts, C would mean “to cooperate” and D “not-to-cooperate” or to defect. For two interacting spins, the energy depends on the spins’ orientations. In games, an agent engaging in a game will receive a payoff depending on both the actions of the agent and his opponent. A two-person game is characterized by a payoff matrix of the form:

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} R & S \\ T & P \end{pmatrix} \end{array}, \qquad (2)$$

where the elements give the payoff to the agent using the action specified by the left-hand column when the opponent uses the action specified by the top row. Different rankings of the elements refer to different games: $T > R > P > S$ for the prisoner’s dilemma, $T > R > S > P$ for the snowdrift game and $R > T > P > S$ for the stag hunt game. In a spin system, interaction takes place in a pairwise manner between neighboring spins. For a given temperature, the interaction energy competes with the thermal energy to arrive at a certain degree of magnetization given by the difference between the number of +1 and -1 spins. In a population, an agent interacts with his neighbours in a pairwise manner defined through their proximity in a lattice or network. Cooperation could emerge when agents compete repeatedly and have an adaptive mechanism to switch from one action to another so as to turn the competing environment to their advantage.

A self-questioning adaptive action updating mechanism provides the bridge to the mapping. Consider an agent i targeted for an action update. As the key quantity is the equivalent neighbouring-spin interaction energy, we consider a neighbour j of agent i. For a pair, the payoff W_{s_i,s_j} to agent i can be read out from Equation (2). The evolutionary dynamics is driven by the desire of improving one's competing environment through the question "Would I be better off had I used the opposite action?" that the agent i asks himself/herself. The payoff for comparison is then given by $W_{\bar{s}_i,s_j}$ with $\bar{s}_i$ being the action opposite than that of s_i. Action updating is carried out as follows. If $W_{\bar{s}_i,s_j} - W_{s_i,s_j} > 0$, using $\bar{s}_i$ in the next round will certainly be beneficial assuming that the opponent retains the action. In this case, agent i will definitely switch action from s_i to $\bar{s}_i$. This could be switching from C to D or from D to C. If $W_{\bar{s}_i,s_j} - W_{s_i,s_j} \leq 0$, switching action will not lead to an immediate advantage. However, an agent may hope for a long-term benefit despite an immediate loss by accepting a switching in action to $\bar{s}_i$ with the probability $e^{(W_{\bar{s}_i,s_j} - W_{s_i,s_j})/K}$, which is a decreasing function of the difference $(W_{\bar{s}_i,s_j} - W_{s_i,s_j})$. Here, K is a noise parameter mimicking an agent's tolerance for immediate losses. This self-questioning adaptive mechanism resembles that of the best-response rule in economic behavior [4] as well as the standard procedure in carrying out Monte Carlo simulations on the Ising model, where K plays the role of a thermal energy related to the temperature through the Boltzmann constant.

The remaining question is whether a competing game with the self-questioning adaptive mechanism could be described by a spin model as that in Equation (1) with suitably chosen parameters J_{ij}. Making the analogy between C (D) and $+(-)$, the adaptive mechanism that accepts an action switching based on the difference $(W_{\bar{s}_i,s_j} - W_{s_i,s_j})$ resembles that of accepting a spin-flip in Monte Carlo simulations based on $(E_{+-} - E_{--})$ or $(E_{-+} - E_{++})$. Note that acceptance in spin-flips is biased towards lowering of energy while acceptance in action-switching is biased towards increasing the payoff. Thus, we have the information that $E_{DC} - E_{CC} = -(W_{DC} - W_{CC}) = R - T$ and $E_{CD} - E_{DD} = -(W_{CD} - W_{DD}) = P - S$. Here, E_{CD} is analogous to E_{+-}, etc. In addition, we expect $E_{CD} = E_{DC}$ when mapped to a spin model. So we have the same situation as discussed in the Introduction for determining J_{ij}. As before, we can make a choice to set a referencing energy $0 = E_{CD} = E_{DC}$, so that $E_{CC} = T - R$ and $E_{DD} = S - P$.

Finally, expressing the results in terms of J_{ij}, we have $J_{++} = J_{CC} = T - R$, $J_{--} = J_{DD} = S - P$, and $J_{+-} = J_{CD} = J_{DC} = J_{-+} = 0$. Note that the relations do not depend on the ranking of the payoff elements and thus the type of the games.

For simplicity, we considered only a single pair of agents in establishing the relationships between J_{ij} and the payoff parameters. The results can be readily extended to structured populations in a lattice or a network of uniform degree. In these cases, an agent i competes with a number q of neighbours and obtains *an averaged payoff* from the games with each of the neighbours. The comparison should then be made between the *averaged payoffs* to the agent when s/he uses the action s_i and $\bar{s}_i$, respectively. For the evolutionary game played on a one-dimensional chain of agents with periodic boundary condition, each agent has $q = 2$ neighbours and the system can be described by the Hamiltonian:

$$
\begin{aligned}
H &= \frac{1}{2}\sum_i \frac{1}{2}\left[J_{++}\delta_{s_i,+}\left(\delta_{s_{i+1},+}+\delta_{s_{i-1},+}\right)+J_{--}\delta_{s_i,-}\left(\delta_{s_{i+1},-}+\delta_{s_{i-1},-}\right)\right] \\
&= \sum_i \frac{1}{2}\left[J_{++}\delta_{s_i,+}\delta_{s_{i+1},+}+J_{--}\delta_{s_i,-}\delta_{s_{i+1},-}\right]
\end{aligned}
\quad (3)
$$

This form stresses that the interaction energy is $(T-R)$ for two neighbouring $++$ spins, it is $(S-P)$ for two neighbouring $--$spins, and zero otherwise. For the system in Equation (3) at equilibrium at a temperature characterized by the thermal energy K, one can look up the exact solution to the magnetization in the Ising model from textbooks [3]. The net magnetization is related to the difference between + spins and – spins, and the result can therefore be translated into an exact expression for the fraction of cooperators in the evolutionary game. The result is:

$$
f_C = \frac{1}{2}\left(\frac{a-b}{\sqrt{(a-b)^2+4}}+1\right), \quad (4)
$$

where $a = e^{-J_{++}/(2K)} = e^{-(T-R)/(2K)}$ and $b = e^{-J_{--}/(2K)} = e^{-(S-P)/(2K)}$. A few remarks follow. While evolutionary games on a chain can conveniently be studied numerically and analytically by the site and link approximations, the exact result in Equation (4) was not given until the mapping to spin model was established [1]. In addition, it was found that the link approximation is an exact approach for games on a chain, but its exactness is far from being obvious without comparing results with Equation (4). Equations (3) and (4) work for any games on a chain, as there is no restriction on the ordering of the payoff elements. Finally, Equation (4) is also the exact solution to the Ising Hamiltonian:

$$
H = \frac{1}{q}\sum_{<ij>}\alpha s_i s_j + \gamma\sum_i s_i \quad , \quad (5)
$$

when it is solved on a chain. Here, $\alpha = [(T-R)-(P-S)]/4$ and $\gamma = [(T-R)+(P-S)]/4$ are combinations of the payoff elements, and q is the number of nearest neighbours in the lattice and the summation is over neighbouring agents i and j. The fact that Equation (5) on a chain and Equation (3) represent the same system should not be surprising. Equation (3) has different energies for neighbouring $++$ spins and $--$spins. This difference can be regarded as an effect due to an external field as represented by γ in Equation (5), while making all neighbouring spins interact with an energy α. Equation (5) is the mapping introduced in Ref.[1]. The discussion that has led to Equation (3) for the special case of a chain can be extended to other lattices and thus it represents an alternative way of establishing a connection between the big research areas of evolutionary games and the Ising model.

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